Some Copositive Thoughts

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Following the suggestion of Manuel et al.:

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Think co(mpletely)positive ! Matrix properties, examples and a clustered bibliography on copositive optimization

Immanuel M. Bomze · Werner Schachinger · Gabriele Uchida

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A is **SPN** if A is a **s**um of a **p**ositive semidefinite matrix and a **n**onnegative one.

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For $n \geq 5$ there exist $n \times n$ copositive matrices that are not SPN.

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Theorem

A is COP $\implies \lambda_1 = \rho$.

Haynsworth & Hoffman (1969)

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A is COP, nonsingular & $\exists t \geq 0$ s.t. $A - tA^3$ is PSD $\implies$ A^{-1} is COP.

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($A^\dagger$ = the Moore-Penrose generalized inverse of A.)

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Condition J for A indefinite, COP: All negative e-values of A are $= -\rho$.

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A is COP, $-\rho$ is an e-value, x is an e-vector corresponding to $-\rho$
$\implies$ $|x|$ is a (nonnegative!) e-vector corresponding to ρ.

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A vector x is **balanced** if $x \perp |x|$.
A subspace $U \subseteq \mathbb{R}^n$ is **balanced** if all vectors in U are balanced.

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A key to the proof, and to constructing all such matrices: understanding balanced subspaces.

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3. $\dim U \leq \frac{n}{2}$.

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Main argument: $x \in \mathbb{R}^n$, $x \geq 0 \implies \|Qx\| \geq \|Px\|$.

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$r = 0$ is possible!

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Any orthogonal COP $A \neq I$ is of this form.

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The idea (for even n):

Let S be $\frac{n}{2} \times k$ with rank $S = k$; $L = \begin{pmatrix} S \\ -S \end{pmatrix}$.

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Note (2): For $k \geq 2$ the generated COP matrices are not necessarily SPN.

Thank you!

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Happy Birthday Manuel!

Many more (co)positive thoughts

(and some balance too)!