P. Amaral
Department of Mathematics,
University Nova de Lisboa

60th birthday of Immanuel Bomze
Optimization, Game Theory, and Data Analysis

Copositivity in fractional optimization:
The testimony of a copositive friendship

FACULDADE DE CIÊNCIAS E TECNOLOGIA
UNIVERSIDADE NOVA DE LISBOA

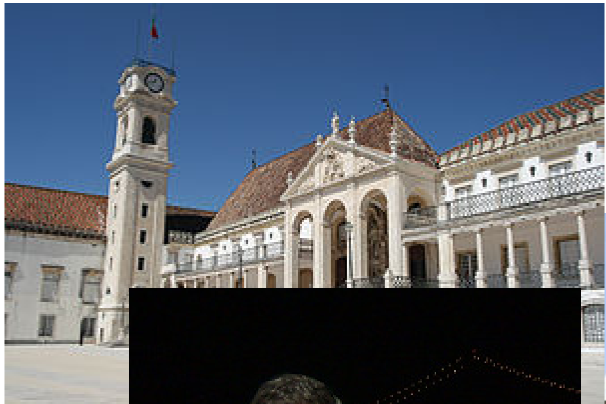
cma FCT
centro de matemática e aplicações

OUTLINE

- TWO EXAMPLES OF DIFFICULT FRACTIONAL PROBLEMS
    - INFEASIBILITY ANALYSIS
    - LINEAR DISCRIMINANT ANALYSIS FOR INTERVAL AND HISTOGRAM DATA
- COMPLETELY POSITIVE FORMULATIONS FOR GENERAL FRACTIONAL PROBLEMS
- LOWER BOUNDS
- MINMAX FRACTIONAL QUADRATIC PROBLEMS
- CONCLUSIONS

First example: Infeasibility in linear systems

2009 - Coimbra



Production planning problem

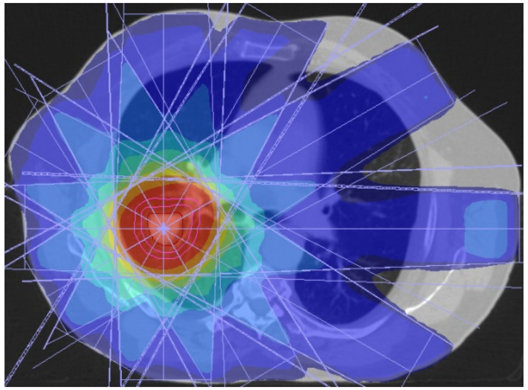
- Amount of products >= contract
- Consuption <= raw materials
- Staff <= availability
- Profit >= minimum
- Costs <= limit

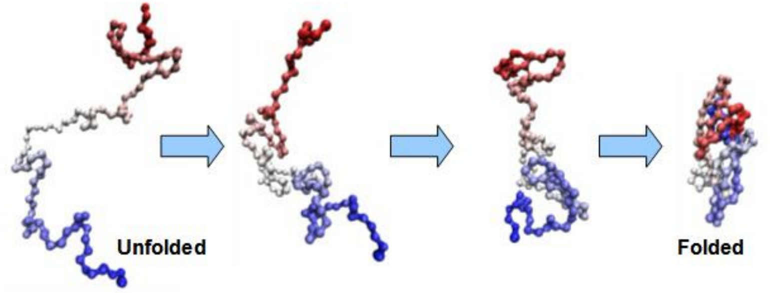
Class timetabling problem

- Schedule classes in the week
- Schedule classes in the day
- Rooms available
- Full professor classes timewindow
- No empty hours

Hard and soft constraints

Radiation Treatment Planning





Protein Folding

Digital Video Broadcasting

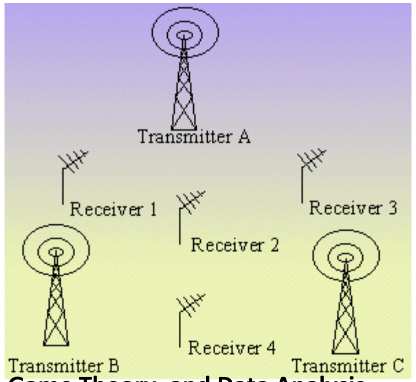
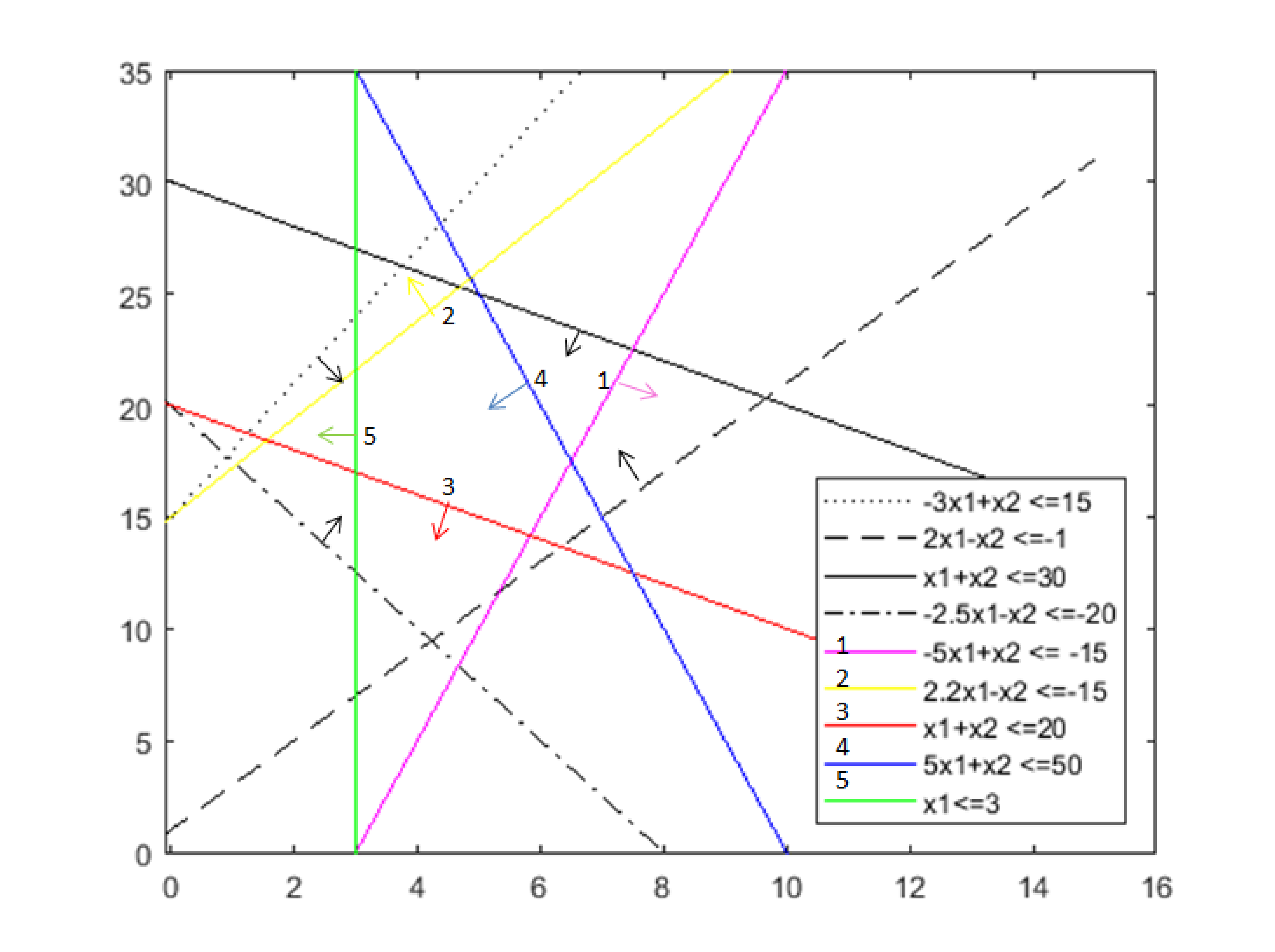


Figure 1: Example of a DVB-T network

$$
\text{hard constraints} \begin{cases} -3x_1 & +x_2 \leq & 15 \\ 2x_1 & -x_2 \leq & -1 \\ x_1 & +x_2 \leq & 30 \\ -2.5x_1 & -x_2 \leq & -20 \end{cases}
$$

$$
\text{soft constraints} \begin{cases} -5x_1 & +x_2 \leq & -15 \\ 2.2x_1 & -x_2 \leq & -15 \\ x_1 & +x_2 \leq & 20 \\ 5x_1 & +x_2 \leq & 50 \\ x_1 & \leq & 3 \end{cases}
$$

$$
x_1, x_2 \geq 0
$$


-3x1+x2 <=15
2x1-x2 <=-1
x1+x2 <=30
-2.5x1-x2 <=-20
1 -5x1+x2 <= -15
2 2.2x1-x2 <=-15
3 x1+x2 <=20
4 5x1+x2 <=50
5 x1<=3
35
30
25
20
15
10
5
0
0
2
4
6
8
10
12
14
16

$$\min \sum_i \sum_j h_{i,j}^2 + \sum p_i^2$$

$$\text{hard constraints} \begin{cases} -3x_1 & +x_2 \leq & 15 \\ 2x_1 & -x_2 \leq & -1 \\ x_1 & +x_2 \leq & 30 \\ -2.5x_1 & -x_2 \leq & -20 \end{cases}$$

$$\text{soft constraints} \begin{cases} -5x_1 & +x_2 \leq & -15 \\ 2.2x_1 & -x_2 \leq & -15 \\ x_1 & +x_2 \leq & 20 \\ 5x_1 & +x_2 \leq & 50 \\ x_1 & \leq & 3 \end{cases}$$

$$x_1, x_2 \geq 0$$

$$\text{hard constraints} \begin{cases} -3x_1 & +x_2 \leq & 15 \\ 2x_1 & -x_2 \leq & -1 \\ x_1 & +x_2 \leq & 30 \\ -2.5x_1 & -x_2 \leq & -20 \end{cases}$$

$$\text{soft constraints} \begin{cases} (-5+h_{11})x_1 & +(1+h_{12})x_2 \leq & -15+p_1 \\ (2.2+h_{21})x_1 & +(-1+h_{22})x_2 \leq & -15+p_2 \\ (1+h_{31})x_1 & +(1+h_{32})x_2 \leq & 20+p_3 \\ (5+h_{41})x_1 & +(1+h_{42})x_2 \leq & 50+p_4 \\ (1+h_{51})x_1 & +(0+h_{52})x_2 \leq & 3+p_5 \end{cases}$$

$$x_1, x_2 \geq 0$$

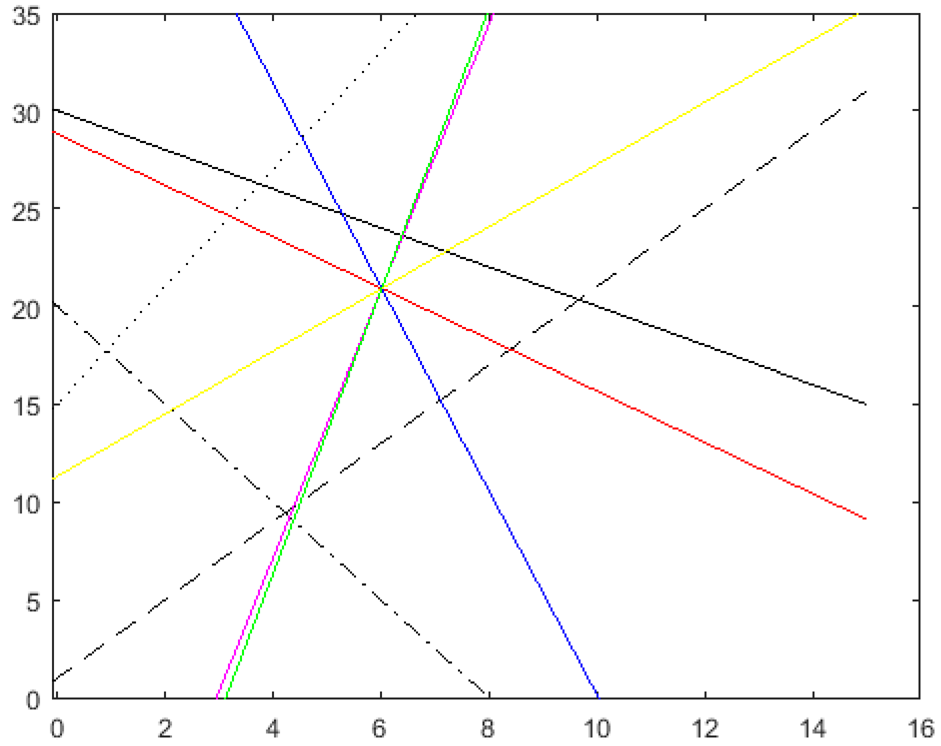
Command Window

Iteration	Open nodes	Total time	Lower bound	Upper bound
1	1	000:00:02	0.00000	0.307097
1	1	000:00:07	0.259773	0.307097
684	322	000:00:37	0.307074	0.307097
1984	859	000:01:08	0.307085	0.307097
3518	1247	000:01:38	0.307089	0.307097
5174	1482	000:02:08	0.307091	0.307097
6959	1157	000:02:38	0.307093	0.307097
8942	960	000:03:09	0.307094	0.307097
11116	612	000:03:39	0.307095	0.307097
13130	0	000:04:01	0.307096	0.307097

```
Cleaning up

                    *** Normal completion ***
```

$$\begin{bmatrix} -0.0739 & -0.2569 & 0.0124 \\ -0.0928 & -0.3231 & 0.0153 \\ -0.0879 & -0.3051 & 0.0149 \\ -0.0125 & -0.0434 & 0.0021 \\ -0.0382 & -0.1330 & 0.0063 \end{bmatrix}$$



$$(P)\ \text{Minimize} \quad \|[H,\, p]\|_F^2$$
$$\text{subject to} \quad (A+H)x \leqslant b+p,$$
$$H \in \mathbb{R}^{m\times n}, \quad p \in \mathbb{R}^m, \quad x \in \mathbf{X}.$$

n+nm+m

$$[H,p] = -\lambda^*[x^{*T}, -1],$$
$$\lambda^* = \frac{1}{1+\|x^*\|^2}(Ax^*-b)^+.$$

n

$$(P)\ \min_{x\in\mathbf{X}} \frac{\|(Ax-b)^+\|^2}{1+\|x\|^2},$$

n+m

$$(\mathrm{P}_F):\quad \phi = \min \quad \frac{\|\mathbf{v}\|^2}{1+\|\mathbf{x}\|^2}$$
$$\text{subject to} \quad A\mathbf{x} - \mathbf{v} \left(\begin{smallmatrix} = \\ \leq \end{smallmatrix}\right) \mathbf{a}$$
$$v_i \geq 0 \text{ for } i = m-r+1, \cdots, m$$
$$\mathbf{x} \in \mathbf{X}.$$

Command Window

```
==========================================================================
  Iteration      Open nodes         Total time     Lower bound       Upper bound
          1               1          000:00:00       0.00000          0.307097
          1               0          000:00:01      0.307096          0.307097

 Cleaning up

                            *** Normal completion ***
```

$$x = \begin{bmatrix} 6.0158 \\ 20.9101 \end{bmatrix}$$

$$v = \begin{bmatrix} 5.8310 \\ 7.3247 \\ 6.9259 \\ 0.9892 \\ 3.0158 \end{bmatrix}$$

Name	m	n
Prob6–Prob10	20	10
Prob11–Prob15	30	15
Prob16–Prob20	40	20

Table 3
Computational results

	RLT-BB						
Problems	ND	CPU	ITER	INITUB	VALOPT	NDOPT	NUPDUB
Galenet	1	0	133	3.7313	3.7313	1	0
Itest2	27	0	751	0.4257	0.4257	1	0
Itest6	1	0	37	82 654 535.9118	82 654 535.9118	1	0
Bgprtr	1	0.01	194	1264.5915	1264.5915	1	0
Forest6	1	0.08	471	3458.7896	3458.7896	1	0
Klein1	221	4.62	23 919	34.6664	34.6664	1	0
Woodinfe	1	0	8193	0.0019	0.0019	1	0
Prob4	5	0	217	157.6815	157.6815	1	0
Prob5	4	0	187	262.8295	262.8295	1	0
Prob6	5	0.01	211	315.1673	315.1673	1	0
Prob7	68	0	1196	214.8726	214.8726	1	0
Prob8	1	0	103	2187.0132	2187.0132	1	0
Prob9	20	0.01	522	149.3484	149.3484	1	0
Prob10	29	0	604	250.8560	250.8560	1	0
Prob11	26	0.01	500	396.0405	396.0405	1	0
Prob12	3	0.01	184	1130.5302	1130.5302	1	0
Prob13	12	0.02	489	679.4526	679.4526	1	0
Prob14	6	0.02	382	756.9513	756.9513	1	0
Prob15	476	1.08	23 552	365.3541	364.4484	424	13
Prob16	26	0.12	2052	1121.5359	1121.5359	1	0
Prob17	27	0.13	1980	1092.9802	1092.9802	1	0
Prob18	9	0.04	623	1750.9055	1738.2852	8	4
Prob19	25	0.17	1999	1104.5614	1104.5614	1	0
Prob20	199	1.26	18 410	944.7827	944.7827	1	0

Table 1: Copositive Relaxation versus `Gloptipoly 3` and `BARON`

Instance	Cop R	Time1(s)	Gap	GPM R	Time2(s)	St.	root B.
ABJ5_0				-0.5275	1.045e+00	1	-26.1028
ABJ5_1				-0.5414	1.014e+00	1	-11.8308
ABJ5_2				-0.5089	9.672e-01	1	-11.9631
ABJ5_3				-0.2207	1.310e+00	1	-3.9613
ABJ5_4				-0.9428	9.984e-01	1	-0.8123
ABJ5_5				+0.2225	1.108e+00	1	-2.2940
ABJ5_6				-0.3671	9.828e-01	1	-8.6291
ABJ5_7				-0.0657	8.892e-01	1	-5.1034
ABJ5_8				-0.3708	9.516e-01	1	-0.4031
ABJ5_9				-0.5753	6.708e-01	1	-0.4553
ABJ10_0				-0.1962	7.010e+02	1	-23.9325
ABJ10_1				-0.4882	5.737e+02	1	-0.4587
ABJ10_2				+0.4288	6.395e+02	1	-3.4076
ABJ10_3				-0.1840	6.298e+02	1	-12.3357
ABJ10_4				-0.2689	5.122e+02	1	-0.2635
ABJ10_5				-0.6198	6.619e+02	1	-55.5414
ABJ10_6				-0.8749	7.123e+02	1	-0.8265
ABJ10_7				-0.0760	6.219e+02	1	-25.4559
ABJ10_8				-0.4558	6.239e+02	1	-0.5056
ABJ10_9				-0.1794	6.183e+02	1	-0.1919
ABJ50_0				O of M	-		-502.4740
ABJ50_1				O of M	-		-0.7856
ABJ50_2				O of M	-		-0.6135
ABJ50_3				O of M	-		-1463.1800
ABJ50_4				O of M	-		-451.7790
ABJ50_5				O of M	-		-0.3962
ABJ50_6				O of M	-		-989.5200
ABJ50_7				O of M	-		-0.4914
ABJ50_8				O of M	-		-490.0360
ABJ50_9				O of M	-		-626.8870
ABJ80_0				O of M	-		-1394.8500
ABJ80_1				O of M	-		-0.4472
ABJ80_2				O of M	-		-0.6681
ABJ80_3				O of M	-		-1849.5000
ABJ80_4				O of M	-		-0.6528
ABJ80_5				O of M	-		-0.3511
ABJ80_6				O of M	-		-2488.4700
ABJ80_7				O of M	-		-1487.1000
ABJ80_8				O of M	-		-736.0130
ABJ80_9				O of M	-		-0.5177

Second example:

Linear Discriminant Analysis for Interval and Histogram Data

Paula Brito , Sónia Dias

NEWFRIENDS=FRIENDS(Immanuel)

$$\max_{x \in \mathbb{R}^{n+}} \frac{x^T B x}{x^T W x}$$

Command Window

```
B =

    0.0035   -0.0032    0.0012   -0.0004   -0.0017    0.0026   -0.0013    0.0018
   -0.0032    0.0035   -0.0004    0.0012    0.0026   -0.0017    0.0018   -0.0013
    0.0012   -0.0004    0.0012    0.0007    0.0005    0.0017    0.0001    0.0011
   -0.0004    0.0012    0.0007    0.0012    0.0017    0.0005    0.0011    0.0001
   -0.0017    0.0026    0.0005    0.0017    0.0027   -0.0002    0.0018   -0.0004
    0.0026   -0.0017    0.0017    0.0005   -0.0002    0.0027   -0.0004    0.0018
   -0.0013    0.0018    0.0001    0.0011    0.0018   -0.0004    0.0013   -0.0005
    0.0018   -0.0013    0.0011    0.0001   -0.0004    0.0018   -0.0005    0.0013

>> W

W =

    0.0553   -0.0527    0.0563   -0.0541   -0.0511    0.0527   -0.0565    0.0580
   -0.0527    0.0553   -0.0541    0.0563    0.0527   -0.0511    0.0580   -0.0565
    0.0563   -0.0541    0.0662   -0.0590   -0.0544    0.0559   -0.0594    0.0623
   -0.0541    0.0563   -0.0590    0.0662    0.0559   -0.0544    0.0623   -0.0594
   -0.0511    0.0527   -0.0544    0.0559    0.0557   -0.0507    0.0565   -0.0517
    0.0527   -0.0511    0.0559   -0.0544   -0.0507    0.0557   -0.0517    0.0565
   -0.0565    0.0580   -0.0594    0.0623    0.0565   -0.0517    0.0671   -0.0604
    0.0580   -0.0565    0.0623   -0.0594   -0.0517    0.0565   -0.0604    0.0671
```

Command Window

```
This BARON run may utilize the following subsolver(s)
For LP:  COIN LP
For NLP: COIN IPOPT with MUMPS and METIS
===========================================================================
Preprocessing found feasible solution with value -.388222693216
Doing local search
Preprocessing found feasible solution with value -.611023475810
Solving bounding LP
Starting multi-start local search
Done with local search
===========================================================================
 Iteration     Open nodes        Total time     Lower bound       Upper bound
         1              1         000:00:01    -0.100000E+52      -0.611023
===========================================================================
User did not provide appropriate variable bounds.
Some model expressions are unbounded.
We may not be able to guarantee globality.
Number of missing variable or expression bounds =      1
Number of variable or expression bounds autoset =      1
===========================================================================
         1              1         000:00:02    -0.100000E+09      -0.611023
      1000            514         000:00:30    -0.100000E+09      -0.611023

Cleaning up

              *** Max. allowable BaR iterations reached ***
```

Size 8x8

UB	Sol	
1,00E+09	6,17E-02	
1,00E+09	1,47E-01	
1,00E+09	2,56E-01	
1,00E+09	1,53E-01	
1,00E+09	1,19E-01	
1,00E+09	1,91E-01	
1,00E+09	2,24E-01	
1,00E+09	2,42E-01	
1,00E+09	1,32E-01	
1,00E+09	3,46E-01	

Fractional Quadratic Problems

$$\psi = \min\left\{ f(\mathbf{x}) = \frac{\mathbf{x}^\top C\mathbf{x} + 2\mathbf{c}^\top \mathbf{x} + \gamma}{\underbrace{\mathbf{x}^\top B\mathbf{x} + 2\mathbf{b}^\top \mathbf{x} + \beta}_{p(x)}} : \underbrace{\mathbf{x} \in \mathbb{R}^n_+ : A\mathbf{x} = \mathbf{a}}_{\mathcal{T}} \right\}$$

Compactness of $\mathcal{T}$ and strict positivity of p over this set implies that

$$\psi = \min\left\{ f(\mathbf{x}) = \frac{\mathbf{x}^\top C\mathbf{x} + 2\mathbf{c}^\top \mathbf{x} + \gamma}{p(\mathbf{x})} : \mathbf{x} \in \mathcal{T} \right\}$$

always has an optimal solution (primal attainability).

Conic Optimization

Reformulated as a conic optimization problem

$$\begin{aligned} \min \quad & \langle C, X \rangle \\ \text{s.t.} \quad & \langle A^i, X \rangle = b_i \quad i \in \{1, \ldots, m\} \\ & X \in \mathcal{K} \end{aligned}$$

$$\begin{aligned} x \in \mathcal{K} = R_n^+ \quad & \text{if} \quad x \geq 0 \\ X \in \mathcal{K} \text{ (Positive Semidefinite)} \quad & \text{if} \quad y^T X y \geq 0 \\ X \in \mathcal{K} \text{ (Copositive)} \quad & \text{if} \quad y^T X y \geq 0 \ \forall y \geq 0 \\ X \in \mathcal{K} \text{ (D-Copositive)} \quad & \text{if} \quad y^T X y \geq 0 \ \forall y \in D \\ X \in \mathcal{K} \text{ (Completely Positive)} \quad & \text{if} \quad X = Y^T Y,\ Y \geq 0 \end{aligned}$$

Conic Optimization

Reformulated as a conic optimization problem

$$\begin{array}{ll} \min & \langle C, X \rangle \\ \text{s.t.} & \langle A^i, X \rangle = b_i \quad i \in \{1, \dots, m\} \\ & X \in \mathcal{K} \end{array}$$

$$x^T A x \longrightarrow \langle A, xx^T \rangle \longrightarrow \langle A, X \rangle \qquad X \in \mathcal{C}_n^* \wedge X \text{ is of rank one} = \mathcal{C}_n^{*rk1}$$

$$\mathcal{C}_n^* = \{D \in \mathcal{M}_n : D = YY^\top, Y \text{ an } n \times k \text{ matrix with } Y \geq O\}$$

Relaxation to a more manageable cone – LOWER BOUND

$$\begin{aligned}\psi &= \min\left\{ f(\mathbf{x}) = \frac{\mathbf{x}^\top C\mathbf{x} + 2c^\top \mathbf{x} + \gamma}{\mathbf{x}^\top B\mathbf{x} + 2\mathbf{b}^\top \mathbf{x} + \beta} : A\mathbf{x} = \mathbf{a}, \mathbf{x} \in \mathbb{R}^n_+ \right\} \\ &= \min\left\{ \overline{C} \bullet X : \overline{B} \bullet X = 1, \overline{A} \bullet X = 0, X \in \mathcal{C}^*_{n+1} \right\}.\end{aligned}$$

cone of completely positive matrices

$$\mathcal{C}^*_n = \left\{ D \in \mathcal{M}_n : D = YY^\top, Y \text{ an } n \times k \text{ matrix with } Y \geq O \right\}$$

$$\overline{A} = \begin{bmatrix} \mathbf{a}^\top \mathbf{a} & -\mathbf{a}^\top A \\ -A^\top \mathbf{a} & A^\top A \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} \beta & \mathbf{b}^\top \\ \mathbf{b} & B \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} \gamma & \mathbf{c}^\top \\ \mathbf{c} & C \end{bmatrix}$$

$$\psi = \min\left\{ f(\mathbf{x}) = \underbrace{\frac{\mathbf{x}^\top C\mathbf{x} + 2\mathbf{c}^\top \mathbf{x} + \gamma}{\mathbf{x}^\top B\mathbf{x} + 2\mathbf{b}^\top \mathbf{x} + \beta}}_{p(x)} : \underbrace{\mathbf{x} \in \mathbb{R}^n_+ : A\mathbf{x} = \mathbf{a}}_{\mathcal{T}} \right\}$$

Compactness of $\mathcal{T}$ and strict positivity of p over this set implies that

$$\psi = \min\left\{ f(\mathbf{x}) = \frac{\mathbf{x}^\top C\mathbf{x} + 2\mathbf{c}^\top \mathbf{x} + \gamma}{p(\mathbf{x})} : \mathbf{x} \in \mathcal{T} \right\}$$

always has an optimal solution (primal attainability).

$$\psi = \min \left\{ f(\mathbf{x}) = \frac{\mathbf{x}^\top C\mathbf{x} + 2\mathbf{c}^\top \mathbf{x} + \gamma 1}{\mathbf{x}^\top B\mathbf{x} + 2\mathbf{b}^\top \mathbf{x} + \beta 1} : \mathbf{x} \in \mathbb{R}^n_+ : A\mathbf{x} = \mathbf{a}1 \right\}$$

$$z = \begin{bmatrix} 1 \\ \mathbf{x} \end{bmatrix}$$

$$\overline{A} = \begin{bmatrix} \mathbf{a}^\top \mathbf{a} & -\mathbf{a}^\top A \\ -A^\top \mathbf{a} & A^\top A \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} \beta & \mathbf{b}^\top \\ \mathbf{b} & B \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} \gamma & \mathbf{c}^\top \\ \mathbf{c} & C \end{bmatrix}$$

$$A\mathbf{x} = \mathbf{a} \Leftrightarrow [-\mathbf{a}\,,\ A]\mathbf{z} = \mathbf{o} \Leftrightarrow \mathbf{z}^\top \overline{A}\mathbf{z} = 0$$

$$\psi = \min\left\{ f(\mathbf{x}) = \frac{\mathbf{x}^\top C\mathbf{x} + 2\mathbf{c}^\top\mathbf{x} + \gamma}{\mathbf{x}^\top B\mathbf{x} + 2\mathbf{b}^\top\mathbf{x} + \beta} : \mathbf{x} \in \mathbb{R}^n_+ : A\mathbf{x} = \mathbf{a} \right\}$$

$$\overline{A} = \begin{bmatrix} \mathbf{a}^\top\mathbf{a} & -\mathbf{a}^\top A \\ -A^\top\mathbf{a} & A^\top A \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} \beta & \mathbf{b}^\top \\ \mathbf{b} & B \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} \gamma & \mathbf{c}^\top \\ \mathbf{c} & C \end{bmatrix}.$$

$$\psi = \min\left\{ \frac{\mathbf{z}^\top\overline{C}\mathbf{z}}{\mathbf{z}^\top\overline{B}\mathbf{z}} : \mathbf{z} \in \mathbb{R}^{n+1}_+ , \; z_1 = 1 , \; \mathbf{z}^\top\overline{A}\mathbf{z} = 0 \right\} .$$

$$\psi = \min \left\{ f(\mathbf{x}) = \underbrace{\frac{\mathbf{x}^\top C\mathbf{x} + 2\mathbf{c}^\top \mathbf{x} + \gamma}{\mathbf{x}^\top B\mathbf{x} + 2\mathbf{b}^\top \mathbf{x} + \beta}}_{p(x)} : \underbrace{\mathbf{x} \in \mathbb{R}^n_+ : A\mathbf{x} = \mathbf{a}}_{\mathcal{T}} \right\}$$

$$\psi = \min \left\{ \frac{\mathbf{z}^\top \overline{C}\mathbf{z}}{\mathbf{z}^\top \overline{B}\mathbf{z}} : \mathbf{z} \in \mathbb{R}^{n+1}_+ , \ z_1 = 1 , \ \mathbf{z}^\top \overline{A}\mathbf{z} = 0 \right\} .$$

$$Z = \mathbf{z}\mathbf{z}^\top$$

$$\mathbf{z}^\top \overline{A}\mathbf{z} = \overline{A} \bullet Z$$

$$\overline{A} , \ \text{psd} \ \ Z_{11} = z_1^2 \text{ and } \mathbf{z} \in \mathbb{R}^{n+1}_+$$

$$\psi = \min \left\{ \frac{\overline{C} \bullet Z}{\overline{B} \bullet Z} : Z_{11} = 1 , \ \overline{A} \bullet Z = 0 , \ \operatorname{rank}(Z) = 1 , \ Z \in \mathcal{C}^*_{n+1} \right\}$$

$$\psi = \min \left\{ \frac{\overline{C} \bullet Z}{\overline{B} \bullet Z} : Z_{11} = 1 \,,\ \overline{A} \bullet Z = 0 \,,\ \mathrm{rank}(Z) = 1 \,,\ Z \in \mathcal{C}^*_{n+1} \right\}$$

$$Z_{11} = 1 \leftarrow Z_{11} > 0$$

.

$$X = \frac{1}{\overline{B} \bullet Z} Z \in \mathcal{C}^*_{n+1}$$

which also has rank one with $X_{11} > 0$ and satisfies

$$\overline{B} \bullet X = 1$$

$$\psi = \min \{ \overline{C} \bullet X : \overline{B} \bullet X = 1, \overline{A} \bullet X = 0, \mathrm{rank}(X) = 1, X_{11} > 0, X \in \mathcal{C}^*_{n+1} \}$$

$$\min \ \left\{\overline{C} \bullet X : \overline{B} \bullet X = 1, \ \overline{A} \bullet X = 0, \ \mathrm{rank}(X) = 1, \right.$$
$$\left. X_{11} > 0, \ X \in \mathcal{C}^*_{n+1}\right\}.$$

$$\min \left\{\overline{C} \bullet X : \overline{B} \bullet X = 1, \overline{A} \bullet X = 0, X \in \mathcal{C}^*_{n+1}\right\}.$$

Lemma 2 *Under the model assumptions* (10),

$$\left\{X \in \mathcal{C}^*_{n+1} : \overline{B} \bullet X = 1, \overline{A} \bullet X = 0\right\}$$
$$= \ conv \left\{\mathbf{z}\mathbf{z}^\top : \mathbf{z} \in \mathbb{R}^{n+1}_+ : z_1 > 0 \, , \ \mathbf{z}^\top \overline{B}\mathbf{z} = 1 \, , \ \overline{A}\mathbf{z} = \mathbf{o}\right\}.$$

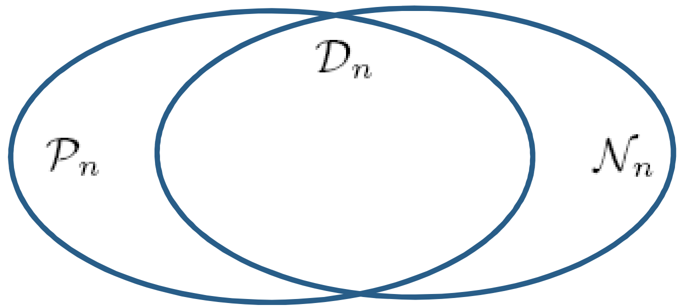
$$\left.\begin{array}{l} \mathcal{T} = \left\{\mathbf{x} \in \mathbb{R}^n_+ : A\mathbf{x} = \mathbf{a}\right\} \neq \emptyset \, ; \\ \ker A \cap \mathbb{R}^n_+ = \{\mathbf{o}\} \Longleftrightarrow A\mathbf{y} \neq \mathbf{o} \text{ if } \mathbf{y} \in \mathbb{R}^n_+ \setminus \{\mathbf{o}\} \, ; \\ \overline{B} \text{ is strictly } \Gamma_{\overline{A}}\text{-copositive: } \ \mathbf{z}^\top \overline{B}\mathbf{z} > 0 \text{ if } \overline{A}\mathbf{z} = \mathbf{o} \, , \ \mathbf{z} \in \mathbb{R}^n_+ \setminus \{\mathbf{o}\}. \end{array}\right\} \quad (10)$$

$$\Gamma_{\overline{A}} = \left\{\mathbf{z} \in \mathbb{R}^{n+1}_+ : \overline{A}\mathbf{z} = \mathbf{o}\right\}.$$

$$\psi = \min \left\{ \overline{C} \bullet X : \overline{B} \bullet X = 1,\ \overline{A} \bullet X = 0,\ \operatorname{rank}(X) = 1, \right.$$
$$\left. X_{11} > 0,\ X \in \mathcal{C}^*_{n+1} \right\}. \tag{13}$$

$$\min \left\{ \overline{C} \bullet X : \overline{B} \bullet X = 1, \overline{A} \bullet X = 0, X \in \mathcal{C}^*_{n+1} \right\}. \tag{14}$$

Theorem 1 *Under the model assumptions (10), problems (13) and (14) are equivalent. Moreover, there is always an optimal solution of the form* $Z^* = Z^*_{11}\mathbf{z}\mathbf{z}^\top$ *to (14) with* $\mathbf{z}^\top = [1\ ,\ (\mathbf{x}^*)^\top]$ *which encodes in* $\mathbf{x}^* \in \mathcal{T}$ *an optimal solution to (4).*

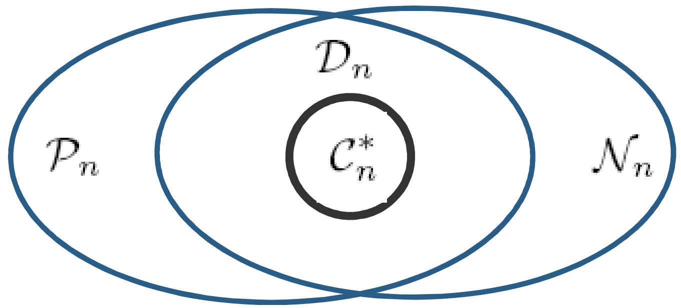


cone of symmetric psd $n \times n$ matrices   cone of nonnegative symmetric matrices

$$\mathcal{P}_n \cap \mathcal{N}_n = \mathcal{D}_n$$

cone of doubly nonnegative matrices

$\mathcal{C}_n^*$ cone of completely positive matrices



$$\mathcal{P}_n \cap \mathcal{N}_n = \mathcal{D}_n$$

cone of doubly nonnegative matrices

Computational Experience

$$
\begin{aligned}
\psi &= \min\left\{ f(\mathbf{x}) = \frac{\mathbf{x}^\top C\mathbf{x} + 2c^\top \mathbf{x} + \gamma}{\mathbf{x}^\top B\mathbf{x} + 2\mathbf{b}^\top \mathbf{x} + \beta} : A\mathbf{x} = \mathbf{a}, \mathbf{x} \in \mathbb{R}^n_+ \right\} \\
&= \min\left\{ \overline{C} \bullet X : \overline{B} \bullet X = 1, \overline{A} \bullet X = 0, X \in \mathcal{C}^*_{n+1} \right\}.
\end{aligned}
$$

Lower bound

$$
\psi_{\text{cop}} = \min\left\{ \overline{C} \bullet X : \overline{B} \bullet X = 1, \overline{A} \bullet X = 0, X \in \mathcal{D}_{n+1} \right\},
$$

Instances Generation

Size n ⟶ n= 4,9,49,79

$$\begin{aligned}\psi &= \min\left\{f(\mathbf{x}) = \frac{\mathbf{x}^\top C\mathbf{x} + 2c^\top\mathbf{x} + \gamma}{\mathbf{x}^\top B\mathbf{x} + 2\mathbf{b}^\top\mathbf{x} + \beta} : A\mathbf{x} = \mathbf{a}, \mathbf{x} \in \mathbb{R}^n_+\right\} \\ &= \min\left\{\overline{C} \bullet X : \overline{B} \bullet X = 1, \overline{A} \bullet X = 0, X \in \mathcal{C}^*_{n+1}\right\}.\end{aligned}$$

Size (n+1,n+1)

$$\psi_{\mathrm{cop}} = \min\left\{\overline{C} \bullet X : \overline{B} \bullet X = 1, \overline{A} \bullet X = 0, X \in \mathcal{D}_{n+1}\right\},$$

SDP size ⟶ 5, 10, 50, 80

Compare de LB obtained by the SDP relaxation with:

Gloptipoly 3 – Generalized Problem of Moments - (rational polynomial optimization problem over a semialgebraic set formulated as linear moment problem)

Baron - Global Optimization code - Branch and Reduce Optimization Navigator – LOWER BOUND AT ROOT, Optimal value for GAPS

Computational experience

	GPM (Gloptipoly3)	Copositive Relaxation	Copositve Relaxation with Reduction
ABJ5_0	eqs m = 210, order n = 98, dim = 2380, blocks = 7 nnz(A) = 2385 + 0, nnz(ADA) = 44100, nnz(L) = 22155 Detailed timing (sec) Pre IPM Post 7.001E-03 3.920E-01 2.002E-03	eqs m = 15, order n = 33, dim = 53, blocks = 3 nnz(A) = 55 + 0, nnz(ADA) = 225, nnz(L) = 120 Detailed timing (sec) Pre IPM Post 5.200E-02 7.001E-02 9.958E-04	eqs m = 6, order n = 27, dim = 33, blocks = 3 nnz(A) = 121 + 0, nnz(ADA) = 36, nnz(L) = 21 Detailed timing (sec) Pre IPM Post 4.003E-03 3.800E-02 9.958E-04
ABJ10_0	eqs m = 5005, order n = 718, dim = 85638, blocks = 12 nnz(A) = 138325 + 0, nnz(ADA) = 25050025, nnz(L) = 12527515 Detailed timing (sec) Pre IPM Post 8.460E-01 4.794E+02 2.800E-02	eqs m = 55, order n = 113, dim = 203, blocks = 3 nnz(A) = 210 + 0, nnz(ADA) = 3025, nnz(L) = 1540 Detailed timing (sec) Pre IPM Post 6.100E-02 6.500E-02 1.006E-03	eqs m = 15, order n = 99, dim = 119, blocks = 3 nnz(A) = 1291 + 0, nnz(ADA) = 225, nnz(L) = 120 Detailed timing (sec) Pre IPM Post 2.900E-02 5.301E-02 1.992E-03

Table 1: Copositive Relaxation versus Gloptipoly 3 and BARON

Instance	Cop R	Time1(s)	Gap	GPM R	Time2(s)	St.	root B.
ABJ5_0	-0.7865	2.700e-02	0.4837	-0.5275	1.045e+00	1	-26.1028
ABJ5_1	-0.4923	3.400e-02	1.8293	-0.5414	1.014e+00	1	-11.8308
ABJ5_2	-0.7693	2.700e-02	0.6771	-0.5089	9.672e-01	1	-11.9631
ABJ5_3	-0.3603	2.900e-02	0.9907	-0.2207	1.310e+00	1	-3.9613
ABJ5_4	-1.2562	2.700e-02	0.5467	-0.9428	9.984e-01	1	-0.8123
ABJ5_5	+0.4643	3.000e-02	0.1552	+0.2225	1.108e+00	1	-2.2940
ABJ5_6	-0.5768	3.100e-02	0.5831	-0.3671	9.828e-01	1	-8.6291
ABJ5_7	-0.0815	3.300e-02	15.2108	-0.0657	8.892e-01	1	-5.1034
ABJ5_8	-0.5946	2.600e-02	0.4752	-0.3708	9.516e-01	1	-0.4031
ABJ5_9	-0.8705	3.100e-02	0.9123	-0.5753	6.708e-01	1	-0.4553
ABJ10_0	-0.3095	3.500e-02	0.5090	-0.1962	7.010e+02	1	-23.9325
ABJ10_1	-0.6779	3.100e-02	0.4781	-0.4882	5.737e+02	1	-0.4587
ABJ10_2	+0.4144	3.400e-02	0.0533	+0.4288	6.395e+02	1	-3.4076
ABJ10_3	-0.3105	3.500e-02	1.2843	-0.1840	6.298e+02	1	-12.3357
ABJ10_4	-0.3885	3.900e-02	0.4746	-0.2689	5.122e+02	1	-0.2635
ABJ10_5	-0.7710	4.300e-02	0.2028	-0.6198	6.619e+02	1	-55.5414
ABJ10_6	-1.2861	3.100e-02	0.5562	-0.8749	7.123e+02	1	-0.8265
ABJ10_7	-0.1154	3.900e-02	1.1720	-0.0760	6.219e+02	1	-25.4559
ABJ10_8	-0.6486	3.100e-02	0.2828	-0.4558	6.239e+02	1	-0.5056
ABJ10_9	-0.3070	4.800e-02	0.5997	-0.1794	6.183e+02	1	-0.1919
ABJ50_0	-0.7435	3.238e+00	0.3552	O of M	-		-502.4740
ABJ50_1	-0.9606	2.731e+00	0.2229	O of M	-		-0.7856
ABJ50_2	-0.7844	3.192e+00	0.2786	O of M	-		-0.6135
ABJ50_3	-0.4022	2.983e+00	0.3630	O of M	-		-1463.1800
ABJ50_4	-0.2677	3.001e+00	0.8199	O of M	-		-451.7790
ABJ50_5	-0.6484	2.981e+00	0.6369	O of M	-		-0.3962
ABJ50_6	-0.5760	3.498e+00	0.3702	O of M	-		-989.5200
ABJ50_7	-0.6486	2.993e+00	0.3201	O of M	-		-0.4914
ABJ50_8	-0.5985	3.221e+00	0.3456	O of M	-		-490.0360
ABJ50_9	-0.3730	3.244e+00	0.3215	O of M	-		-626.8870
ABJ80_0	-0.4427	5.049e+01	0.5019	O of M	-		-1394.8500
ABJ80_1	-0.5806	5.532e+01	0.2984	O of M	-		-0.4472
ABJ80_2	-0.8597	5.532e+01	0.2869	O of M	-		-0.6681
ABJ80_3	-0.4345	5.519e+01	0.3302	O of M	-		-1849.5000
ABJ80_4	-0.8625	5.101e+01	0.3214	O of M	-		-0.6528
ABJ80_5	-0.4670	5.117e+01	0.3301	O of M	-		-0.3511
ABJ80_6	-0.3473	5.539e+01	0.6090	O of M	-		-2488.4700
ABJ80_7	-0.5883	5.105e+01	0.3607	O of M	-		-1487.1000
ABJ80_8	-0.4181	5.532e+01	0.5004	O of M	-		-736.0130
ABJ80_9	-0.7023	5.099e+01	0.3568	O of M	-		-0.5177

Linear discriminant problems 8x8

UB	Sol	UB
1,00E+09	6,17E-02	6,17E-02
1,00E+09	1,47E-01	1,52E-01
1,00E+09	2,56E-01	2,60E-01
1,00E+09	1,53E-01	1,56E-01
1,00E+09	1,19E-01	1,20E-01
1,00E+09	1,91E-01	1,91E-01
1,00E+09	2,24E-01	2,35E-01
1,00E+09	2,42E-01	2,42E-01
1,00E+09	1,32E-01	1,32E-01
1,00E+09	3,46E-01	3,47E-01

And minimax fractional problems

$$\min_{x\in\Omega} \max_{i\in I} \frac{f_i(x)}{g_i(x)} .$$

- Allows the decision maker to compute the best strategy under the worst-case scenario.
- The solution value will not deteriorate whichever scenario turns out to be the true one.
- The real solution evaluation will be at least as good as the min-max value.
- Applications of single ratio can be directly extended cases where there is some uncertainty related to outside factors.

Copositive reformulation

$$\text{(MMDFP)} \qquad \min_{\mathsf{x}\in\Omega} \max_{i\in I} \frac{\mathsf{x}^\top \mathsf{Q}_i \mathsf{x} + 2\mathsf{b}_i^\top \mathsf{x} + c_i}{\mathsf{r}_i^\top \mathsf{x} + d_i} .$$

$$\Omega = \left\{ \mathsf{x} \in \mathbb{R}^n_+ : \mathsf{A}\mathsf{x} = \mathsf{a} \, , \; \mathsf{x}^\top \mathsf{A}_q \mathsf{x} + \mathsf{a}_q \mathsf{x} + \alpha_q \leq 0 \text{ for all } q \in [1:p] \right\}$$

We use a Shor lifting: $\mathsf{y}^\top = [1\ \mathsf{x}^\top]$ and for $i \in I$ abbreviate by

$$\widehat{h}(\mathsf{y}) = \max_{i \in I} \frac{\mathsf{y}^\top \widehat{\mathsf{Q}}_i \mathsf{y}}{\widehat{\mathsf{r}}_i^\top \mathsf{y}}$$

with

$$\widehat{\mathsf{Q}}_i = \begin{bmatrix} c_i & \mathsf{b}_i^\top \\ \mathsf{b}_i & \mathsf{Q}_i \end{bmatrix} \text{ and } \widehat{\mathsf{r}}_i = \begin{bmatrix} d_i \\ 2\mathsf{r}_i \end{bmatrix}.$$

Next we :

$$\mathsf{A}\mathsf{x} = \mathsf{a} \quad \Longleftrightarrow \quad \|\mathsf{A}\mathsf{x} - \mathsf{a}\|^2 \leq 0 \quad \Longleftrightarrow \quad \mathsf{y}^\top \widehat{\mathsf{A}_0} \mathsf{y} \leq 0\,,$$

where

$$\widehat{\mathsf{A}_0} = \begin{bmatrix} \mathsf{a}^\top \mathsf{a} & -\mathsf{a}^\top \mathsf{A} \\ -\mathsf{A}^\top \mathsf{a} & \mathsf{A}^\top \mathsf{A} \end{bmatrix}.$$

Likewise, homogenize the quadratic constraints by introducing, for all $q \in [1:p]$,

$$\widehat{\mathsf{A}_q} = \begin{bmatrix} \alpha_q & \mathsf{a}_q^\top \\ \mathsf{a}_q & \mathsf{A}_q \end{bmatrix} .$$

So, denoting

$$\widehat{\Omega} = \left\{ \mathsf{y} \in \mathbb{R}_+^{n+1} : y_0 = 1 \, , \, \mathsf{y}^\top \widehat{\mathsf{A}_q} \mathsf{y} \le 0 \text{ for all } q \in [0:p] \right\} ,$$

we arrive at

$$\lambda^* = \min_{\mathsf{y} \in \widehat{\Omega}} \widehat{h}(\mathsf{y}) = \min_{\mathsf{y} \in \widehat{\Omega}} \max_{i \in I} \frac{\mathsf{y}^\top \widehat{\mathsf{Q}}_i \mathsf{y}}{\widehat{\mathsf{r}}_i^\top \mathsf{y}} . \qquad (12)$$

we introduce another variable $v \in \mathbb{R}$ and obtain

$$\begin{aligned}
\lambda^* &= \min_{\mathsf{y}\in\widehat{\Omega}} \left\{ \max_{i\in I} \frac{\mathsf{y}^\top \widehat{\mathsf{Q}}_i \mathsf{y}}{\widehat{\mathsf{r}}_i^\top \mathsf{y}} \right\} \\
&= \min_{(\mathsf{y},v)\in\widehat{\Omega}\times\mathbb{R}} \left\{ v : \frac{\mathsf{y}^\top \widehat{\mathsf{Q}}_i \mathsf{y}}{\widehat{\mathsf{r}}_i^\top \mathsf{y}} \le v \text{ for all } i \in I \right\} \\
&= \min_{(\mathsf{y},v)\in\widehat{\Omega}\times\mathbb{R}} \left\{ v : \mathsf{y}^\top \widehat{\mathsf{Q}}_i \mathsf{y} \le v\widehat{\mathsf{r}}_i^\top \mathsf{y} \text{ for all } i \in I \right\} .
\end{aligned}$$

Now considering $z = [y^\top \; v]^\top$ and

$$\breve{Q}_i = \begin{bmatrix} \widehat{Q}_i & o \\ o^\top & 0 \end{bmatrix} \quad \text{for } i \in I\,,$$

$$\breve{R}_i = \begin{bmatrix} O & \frac{1}{2}\widehat{r}_i \\ \frac{1}{2}\widehat{r}_i^\top & 0 \end{bmatrix} \quad \text{for } i \in I\,,$$

$$\breve{A}_q = \begin{bmatrix} \widehat{A_q} & o \\ o^\top & 0 \end{bmatrix} \quad \text{for } q \in [0:m]\,,$$

we obtain

$$\lambda^* =$$

$$= \min_{\mathsf{z}\in\mathbb{R}_+^{n+1}} \left\{ z_{n+1} : z_0 = 1\,,\ \mathsf{z}^\top \breve{\mathsf{Q}}_i \mathsf{z} \le \mathsf{z}^\top \breve{\mathsf{R}}_i \mathsf{z},\ i \in I\,, \mathsf{z}^\top \breve{\mathsf{A}}_q \mathsf{z} \le 0\,,\ q\in[0\!:\!p] \right\}.$$

$$= \min_{\mathsf{z}\in\mathbb{R}_+^{n+1}} \left\{ z_{n+1} : z_0 = 1\,,\ \mathsf{z}^\top (\breve{\mathsf{Q}}_i - \breve{\mathsf{R}}_i) \mathsf{z} \le,\ i \in I\,, \mathsf{z}^\top \breve{\mathsf{A}}_q \mathsf{z} \le 0,\ q\in[0\!:\!p] \right\}.$$

$\mathsf{X} = \mathsf{z}\mathsf{z}^\top\ , (z^*_{n+1} \ge 0)$ and with $z^\top W z = W \bullet X$

$$\gamma^* =$$

$$\min_{\mathsf{X}\in\mathcal{C}^{rk1}_{n+2}} \left\{ X_{n+1,n+1} : X_{00} = 1 (\breve{\mathsf{Q}}_i - \breve{\mathsf{R}}_i) \bullet \mathsf{X} \le 0, i \in I, \breve{\mathsf{A}}_q \bullet X \le 0, q\in[0\!:\!p] \right\},$$

where $\mathcal{C}^{rk1}_{n+2} = \{\mathsf{X} \in \mathcal{C}_{n+2} : \text{rank } \mathsf{X} = 1\}$.

Dropping the rank constraint leads to the copositive relaxation

$$\gamma^*_{CP} = \min_{\mathsf{X}\in\mathcal{C}_{n+2}} \left\{ X_{n+1,n+1} : X_{00} = 1, (\breve{\mathsf{Q}}_i - \breve{\mathsf{R}}_i) \bullet \mathsf{X} \leq 0, i \in I, \breve{\mathsf{A}}_q \bullet X \leq 0, \right\}$$

$$q \in [0:p]$$

with its dual

$$\gamma^*_{COP} = \sup_{\mathsf{u}\in\mathbb{R}^{m+q+2}_+} \left\{ u_0 : \mathsf{E}_{n+1} - u_0\mathsf{E}_0 - \sum_{i=1}^{m} u_i(\breve{\mathsf{Q}}_i - \breve{\mathsf{R}}_i) - \sum_{q=0}^{p} \mu_q \breve{\mathsf{A}}_q \in \mathcal{C}^*_{n+2} \right\},$$

where $\mathsf{E}_k \bullet \mathsf{X}_k = X_{kk}$.

Numerical experiments

- PC, Intel(R) Core(TM) i7-2640M, 2.80 Ghz,400 GB RAM.
- `Matlab 2013Ra` was used to run the global optimization solver `BARON`
- $\lambda^* = \min\left\{ v : \frac{f_i(\mathsf{x})}{g_i(\mathsf{x})} \le v, i \in I, \mathsf{Ax} = \mathsf{a}, \mathsf{A}_q \mathsf{x} \le a_q, \mathsf{x} \ge 0, v_l \le v \le v_u \right\}$
- `SDPT3(4.0)/Octave`,
- interface `YALMIP` was used to call SDPT3.
- Test instances were randomly generated. $m = 3, 5, 10$ ratios and $n \in \{5, 25, 50, 75\}$.

$$\text{Gap1}=100\frac{\text{BARON UB}-\text{BARON LB}}{\text{BARON UB}}$$

$$\text{Gap0}=100\frac{\text{BARON UB}-\text{YLB}}{\text{BARON UB}}$$

$$\text{Gap2}=100\frac{\text{BARON UB}-\text{BARON LB}}{\text{BARON UB}}$$

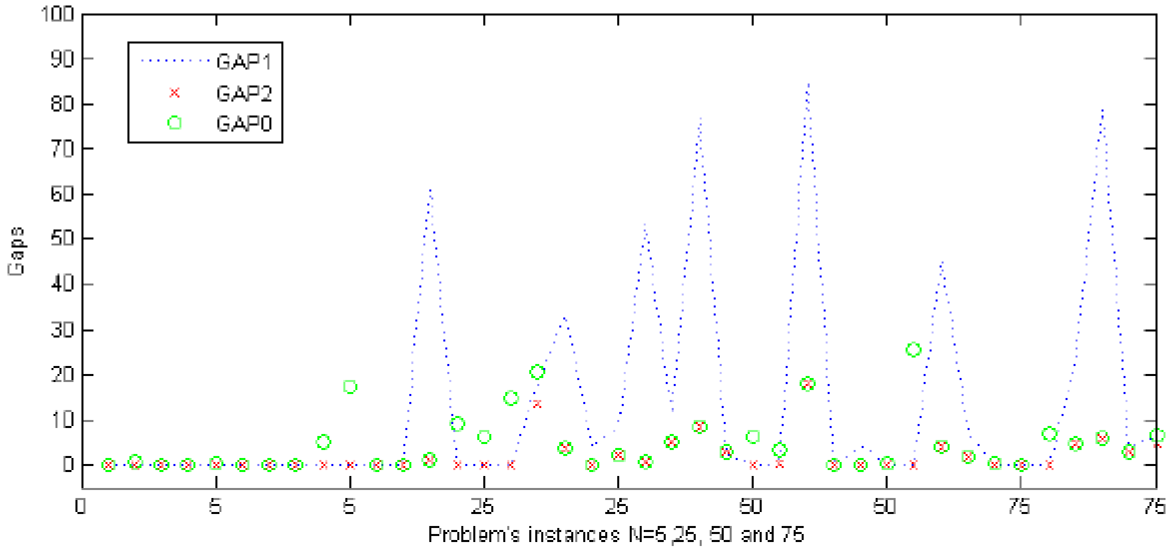


Figure 3: Relative gaps for the $m = 5$ instances

Conclusions

Better than a positive friendship just a copositive one!

Future work?

THANK YOU MANUEL!