Sampling methods for multistage robust convex optimization problems

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Motivation

- In real life applications, several decision problems are **multiperiod** and affected by **uncertainty**;
- Adjustable Robust Counterpart (ARC) have been introduced to extend *static* robust optimization to a **dynamic framework**;
- ARC turns out to be **computationally intractable** in many cases;
- **Constraints sampling** allows to obtain approximate solutions for static robust optimization problems.

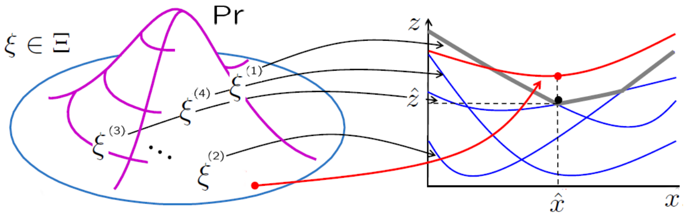
In this talk we consider:

- **probabilistic guarantee** for constraint sampling for **robust convex multistage optimization problems**;
- **Bounds** on the **probability of violation** of the scenario approach are provided and numerical results presented.

Constraint sampling approximation for single stage RO problems

Given:

- $z(x, \xi)$, **convex** in $x \in \mathbb{R}^d$, with $\xi \in \Xi$;
- $\xi^{(1)}, \ldots, \xi^{(N)} \longrightarrow z(x, \xi^{(1)}), \ldots, z(x, \xi^{(N)})$;
- $\min_x \max_{i=1,\ldots,N} z(x, \xi^{(i)})$, solution $\hat{x}$: $\hat{z} = \max_{i=1,\ldots,N} z(x, \xi^{(i)})$.



[Calafiore and Campi (2004) - Campi and Garatti (2008)]

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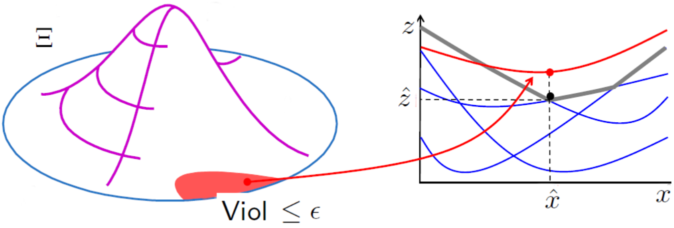

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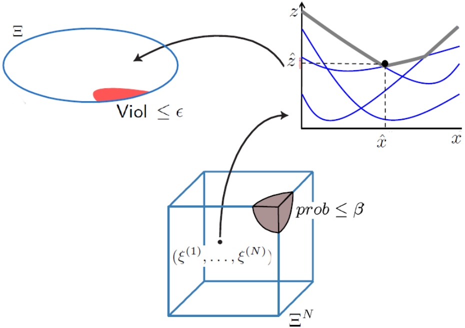
Constraint sampling approximation for single stage RO problems



Theorem (Campi and Garatti (2008))

For any **probability level** $\epsilon \in (0,1)$

$$\Pr(Viol(\hat{x}) > \epsilon) \quad \leq \quad \sum_{k=0}^{d-1} \binom{N}{k} \epsilon^k (1-\epsilon)^{N-k} = B(N,\epsilon,d)$$

Two-stage robust linear problem

We consider the following **two-stage robust linear problem**:

$$\begin{aligned} \mathrm{RO}_2 \quad := \quad & \min_{x^1} c^{1\top} x^1 + \sup_{\xi^1 \in \Xi^1} \left[\min_{x^2(\xi^1)} c^{2\top}(\xi^1)\, x^2(\xi^1) \right] \\ & \text{s.t. } Ax^1 = h^1, \quad x^1 \geq 0 \\ & \qquad T^1(\xi^1)x^1 + W^2(\xi^1)x^2(\xi^1) = h^2(\xi^1), \quad x^2(\xi^1) \geq 0, \quad \forall \xi^1 \in \Xi^1 \end{aligned}$$

- **Nonanticipativity**

  $\text{decision}(x^1) \rightarrow \text{observation}(\xi^1) \rightarrow \text{decision}(x^2)$
- **Adjustable decisions**
  the decisions at stage 2 are functions of ξ^1.
- **Relatively complete recourse**.

The robust problem

Theorem

The robust two-stage linear program RO_2 is equivalent to the following **robust** *problem*

$$
\begin{aligned}
RwC_2 \quad := \quad & \min_{x^1,\gamma} \gamma \\
& s.t.\ Ax^1 = h^1,\ x^1 \geq 0 \\
& \qquad \forall \xi^1 \in \Xi^1,\ \exists x^2(\xi^1) \in \mathbb{R}^{n_2} \quad \text{satisfying} \\
& \qquad {c^1}^\top x^1 + {c^2}^\top(\xi^1)\, x^2(\xi^1) \leq \gamma \\
& \qquad x^2(\xi^1) \geq 0,\ T^1(\xi^1)x^1 + W^2(\xi^1)x^2(\xi^1) = h^2(\xi^1).
\end{aligned}
$$

where we distinguish between **first-stage variables** *(x^1, γ) and* **second-stage** *$x^2(\xi^1)$.*

The scenario problem

- We draw **N samples** of the random variable ξ^1: $\xi^{1^{(1)}}, \ldots, \xi^{1^{(N)}}$;
- For every random sample $\xi^{1^{(i)}}$ we introduce a **second-stage variable** x_i^2.

Scenario problem

$$\mathrm{SwC}_2^N \quad := \quad \min_{x^1, \gamma, x_1^2, \ldots, x_N^2} \gamma$$
$$\text{s.t. } Ax^1 = h^1, \quad x^1 \geq 0$$
$$c^{1^\top} x^1 + c^{2^\top} \left(\xi^{1^{(i)}}\right) x_i^2 \leq \gamma, \; i = 1, \ldots, N,$$
$$T^1(\xi^{1^{(i)}})x^1 + W^2(\xi^{1^{(i)}})x_i^2 = h^2(\xi^{1^{(i)}}), \; x_i^2 \geq 0, \; i = 1, \ldots, N.$$

Proposition

$$SwC_2^N \leq RO_2;$$
$$\lim_{N \to \infty} SwC_2^N = RO_2.$$

Violation probability and probabilistic results

Violation probability

$$\mathrm{V}_2(x^1,\gamma) := \Pr\left\{\xi^1 \in \Xi^1 \text{ for which } \nexists x^2(\xi^1) \in \mathbb{R}_+^{n_2} : \left[\begin{array}{l} {c^1}^\top x^1 + {c^2}^\top(\xi^1)\, x^2(\xi^1) \leq \gamma \\ T^1(\xi^1)x^1 + W^2(\xi^1)x^2(\xi^1) = h^2(\xi^1) \end{array}\right.\right\}.$$

Theorem (two-stage robust linear case)

Given an accuracy level $\epsilon \in (0,1)$, *the solution* $(\hat{x}_N^1, \hat{\gamma}_N)$ *of* SwC_2^N *satisfies*

$$\Pr\left\{V_2(\hat{x}^1,\hat{\gamma})_{SwC_2^N} > \epsilon\right\} \leq B(N,\epsilon,n_1+1) := \sum_{k=0}^{n_1}\binom{N}{k}\epsilon^k(1-\epsilon)^{N-k},$$

The **sample complexity** satisfies:

$$N(\epsilon,\beta) \geq \frac{1}{\epsilon}\frac{e}{e-1}\left(\ln\frac{1}{\beta} + n_1 + 1\right), \quad (e \text{ is the Euler constant})$$

Multistage Linear Robust Problem

$$\mathrm{RO}_H := \min_{\mathbf{x}} \sup_{\boldsymbol{\xi}^{H-1}} z\left[(\mathbf{x})), \boldsymbol{\xi}^{H-1}\right]$$

$$= \min_{x^1} c^{1\top} x^1 +$$

$$+ \sup_{\xi^1 \in \Xi^1} \left[\min_{x^2(\xi^1)} c^{2\top}(\xi^1)\, x^2(\xi^1) + \sup_{\xi^2 \in \Xi^2} \left[\cdots + \sup_{\xi^{H-1} \in \Xi^{H-1}} \left[\min_{x^H(\boldsymbol{\xi}^{H-1})} c^{H\top}(\boldsymbol{\xi}^{H-1})\, x^H(\boldsymbol{\xi}^{H-1}) \right]\right]\right]$$

$$\text{s.t. } Ax^1 = h^1,\ x^1 \geq 0$$

$$T^1(\xi^1)x^1 + W^2(\xi^1)x^2(\xi^1) = h^2(\xi^1),\ \forall \xi^1 \in \Xi^1$$

$$\vdots$$

$$T^{H-1}(\boldsymbol{\xi}^{H-1})x^{H-1}(\boldsymbol{\xi}^{H-2}) + W^H(\boldsymbol{\xi}^{H-1})x^H(\boldsymbol{\xi}^{H-1}) = h^H(\boldsymbol{\xi}^{H-1}),\ \forall \boldsymbol{\xi}^{H-1} \in \Xi$$

$$x^t(\boldsymbol{\xi}^{t-1}) \geq 0,\ t = 2, \ldots, H,\quad \forall \boldsymbol{\xi}^{t-1} \in \times_{\tau=1}^{t-1} \Xi^\tau,$$

where the **nonanticipative** decision process takes the form:

$$\begin{aligned} \text{decision}(x^1) &\to \text{observation}(\xi^1) \to \text{decision}(x^2) \to \text{observation}(\xi^2) \to \ldots \\ \text{decision}(x^{H-1}) &\to \text{observation}(\xi^{H-1}) \to \text{decision}(x^H). \end{aligned}$$

The scenario approach

- We draw N_1 samples of ξ^1: $\xi^{1(i)}, i = 1, \dots, N_1$

$$
\begin{aligned}
\mathrm{SRO}_3^{N_1} \quad := \quad & \min_{x^1, \gamma^1} c^{1\top} x^1 + \gamma^1 \\
& \text{s.t. } A x^1 = h^1, \; x^1 \geq 0 \\
& \qquad \mathcal{Q}_1(x^1, \xi^{1(i)}) \leq \gamma^1, \quad i = 1, \dots, N_1.
\end{aligned}
$$

- We draw N_2 samples of ξ^2: $\xi^{2(j)}, j = 1, \dots, N_2$:

$$
\begin{aligned}
\hat{\mathcal{Q}}_{1,N_2}(x^1, \xi^{1(i)}) \quad := \quad & \min_{x^2, \gamma^2} c^{2\top}(\xi^{1(i)}) x^2(\xi^{1(i)}) + \gamma^2 \\
& \text{s.t. } T^1(\xi^{1(i)}) x^1 + W^2(\xi^{1(i)}) x^2(\xi^{1(i)}) = h^2(\xi^{1(i)}) \\
& \qquad \mathcal{Q}_2(x^2, \xi^{2(j)}) \leq \gamma^2, \quad j = 1, \dots, N_2 \\
& \qquad x^2(\xi^{1(i)}) \geq 0,
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{Q}_2(x^2, \xi^{2(j)}) \quad := \quad & \min_{x^3} c^{3\top}(\xi^{2(j)}) x^3(\xi^{2(j)}) \\
& \text{s.t. } T^2(\xi^{2(j)}) x^2(\xi^{1(i)}) + W^3(\xi^{2(j)}) x^3(\xi^{2(j)}) = h^3(\xi^{2(j)}) \\
& \qquad x^3(\xi^{2(j)}) \geq 0.
\end{aligned}
$$

The scenario approach

- We draw N_1 samples of ξ^1: $\xi^{1(i)}, i = 1, \ldots, N_1$

$$\mathrm{SRO}_3^{N_1} := \min_{x^1,\gamma^1} c^{1\top} x^1 + \gamma^1$$
$$\text{s.t. } Ax^1 = h^1,\; x^1 \geq 0$$
$$\mathcal{Q}_1(x^1, \xi^{1(i)}) \leq \gamma^1, \quad i = 1, \ldots, N_1.$$

- We draw N_2 samples of ξ^2: $\xi^{2(j)}, j = 1, \ldots, N_2$:

$$\widehat{\mathcal{Q}}_{1,N_2}(x^1, \xi^{1(i)}) := \min_{x^2,\gamma^2} c^{2\top}(\xi^{1(i)}) x^2(\xi^{1(i)}) + \gamma^2$$
$$\text{s.t. } T^1(\xi^{1(i)})x^1 + W^2(\xi^{1(i)})x^2(\xi^{1(i)}) = h^2(\xi^{1(i)})$$
$$\mathcal{Q}_2(x^2, \xi^{2(j)}) \leq \gamma^2, \quad j = 1, \ldots, N_2$$
$$x^2(\xi^{1(i)}) \geq 0,$$

with

$$\mathcal{Q}_2(x^2, \xi^{2(j)}) := \min_{x^3} c^{3\top}(\xi^{2(j)}) x^3(\xi^{2(j)})$$
$$\text{s.t. } T^2(\xi^{2(j)})x^2(\xi^{1(i)}) + W^3(\xi^{2(j)})x^3(\xi^{2(j)}) = h^3(\xi^{2(j)})$$
$$x^3(\xi^{2(j)}) \geq 0.$$

The scenario approach

- We draw N_1 samples of ξ^1: $\xi^{1(i)}, i = 1, \dots, N_1$

$$\begin{aligned} \mathrm{SRO}_3^{N_1} \quad := \quad & \min_{x^1, \gamma^1} c^{1\top} x^1 + \gamma^1 \\ & \text{s.t. } A x^1 = h^1, \; x^1 \geq 0 \\ & \quad \mathcal{Q}_1(x^1, \xi^{1(i)}) \leq \gamma^1, \quad i = 1, \dots, N_1. \end{aligned}$$

- We draw N_2 samples of ξ^2: $\xi^{2(j)}, j = 1, \dots, N_2$:

$$\begin{aligned} \hat{\mathcal{Q}}_{1,N_2}(x^1, \xi^{1(i)}) \quad := \quad & \min_{x^2, \gamma^2} c^{2\top}(\xi^{1(i)}) x^2(\xi^{1(i)}) + \gamma^2 \\ & \text{s.t. } T^1(\xi^{1(i)}) x^1 + W^2(\xi^{1(i)}) x^2(\xi^{1(i)}) = h^2(\xi^{1(i)}) \\ & \quad \mathcal{Q}_2(x^2, \xi^{2(j)}) \leq \gamma^2, \quad j = 1, \dots, N_2 \\ & \quad x^2(\xi^{1(i)}) \geq 0, \end{aligned}$$

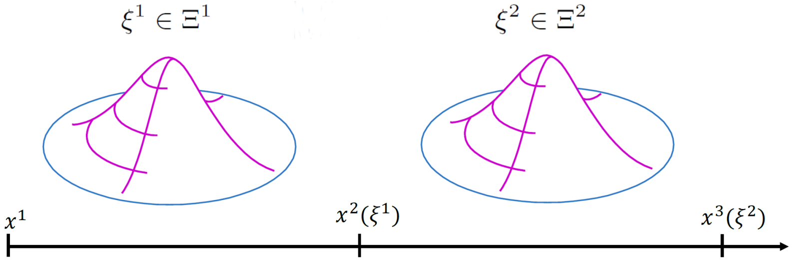
with

$$\begin{aligned} \mathcal{Q}_2(x^2, \xi^{2(j)}) \quad := \quad & \min_{x^3} c^{3\top}(\xi^{2(j)}) x^3(\xi^{2(j)}) \\ & \text{s.t. } T^2(\xi^{2(j)}) x^2(\xi^{1(i)}) + W^3(\xi^{2(j)}) x^3(\xi^{2(j)}) = h^3(\xi^{2(j)}) \\ & \quad x^3(\xi^{2(j)}) \geq 0. \end{aligned}$$

The scenario approach

Scenario problem

$$\mathrm{SRO}_3^{N_1,N_2} \quad := \quad \min_{x^1,\gamma^1} c^{1\top} x^1 + \gamma^1$$
$$\text{s.t. } Ax^1 = h^1,\ x^1 \geq 0$$
$$\widehat{\mathcal{Q}}_{1,N_2}(x^1, \xi^{1(i)}) \leq \gamma^1, \quad i = 1, \ldots, N_1.$$



The scenario approach

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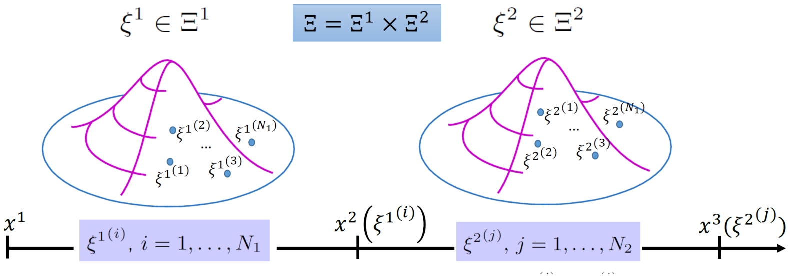


The scenario approach

Definition (violation probability)

$$V_H(x^1, \gamma^1, \gamma^2, \dots, \gamma^{H-1}) :=$$

$$\Pr \left\{ \boldsymbol{\xi}^{H-1} \in \Xi \text{ for which } \quad \nexists\, x^2(\xi^1) \in \mathbb{R}_+^{n_2}, \dots, x^H(\boldsymbol{\xi}^{H-1}) \in \mathbb{R}_+^{n_H} : \left[\begin{array}{l} Ax^1 = h^1 \\ T^1(\xi^1)x^1 + W^2(\xi^1)x^2(\xi^1) = h^2(\xi^1) \\ \vdots \\ T^{H-1}(\boldsymbol{\xi}^{H-1})x^{H-1}(\boldsymbol{\xi}^{H-2}) + W^H \boldsymbol{\xi}^{H-1})x^H(\boldsymbol{\xi}^{H-1}) = h^H(\boldsymbol{\xi}^{H-1}) \\ c^2(\xi^1)x^2(\xi^1) + \gamma^2 \leq \gamma^1 \\ \vdots \\ c^H(\boldsymbol{\xi}^{H-1})x^H(\boldsymbol{\xi}^{H-1}) \leq \gamma^{H-1} \end{array} \right. \right\}.$$

Probabilistic bounds and sample complexity

Theorem (multistage robust linear case)

Given the accuracy levels $\epsilon_1, \epsilon_2, \ldots, \epsilon_{H-1} \in (0,1)$, *and* $\epsilon = \sum_{i=1}^{H-1} \epsilon_i \in (0,1)$, *the solution* $(\hat{x}^1, \hat{\gamma}^1, \hat{\gamma}^2, \ldots, \hat{\gamma}^{H-1})$ *of* $SRO_H^{N_1,N_2,\ldots,N_{H-1}}$ *satisfies*

$$\Pr^N\left\{ V_H(\hat{x}^1, \hat{\gamma}^1, \hat{\gamma}^2, \ldots, \hat{\gamma}^{H-1})_{S_H^{N_1,N_2,\ldots,N_{H-1}}} > \epsilon \right\} \leq \min\left\{ \sum_{t=1}^{H-1} \left(\prod_{i=0}^{t-1} N_i \right) B(N_t, \epsilon_t, n_t + 1), 1 \right\},$$

with $N_0 = 1$ *and* $N = \prod_{t=1}^{H-1} N_t$.

- **Sample complexity**: given $\epsilon_t \in (0,1)$ and $\beta \in (0,1)$:

$$N_t(\epsilon_t, \beta) \geq \frac{1}{\epsilon_t} \frac{e}{e-1} (\ln \frac{1}{\beta} + n_t + 1),$$

and the total number of required scenarios is $N = \prod_{t=1}^{H-1} N_t$.

Differences with other constraint sampling approaches for multistage convex robust problems

Our approach	**Vayanos, Kuhn, Rustem (2012)**
• Scenario approach	• Decision rules approximation
• No explicit knowledge	• Explicit parameterizations
of $x^t(\boldsymbol{\xi}^{t-1})$	of $x^t(\boldsymbol{\xi}^{t-1}) = \sum_{k=1}^{M^t} c_k^t \phi_k^t(\boldsymbol{\xi}^{t-1})$ where $\phi_1^t, \ldots, \phi_{M^t}^t$ are **basis functions** c_k^t the coefficients (**decision variables**)
• Less conservative	• The existence of a solution with pre-specified form
• N_t depends only on the dim. of variables $(n_t + 1)$ $N = \prod_{t=1}^{H-1} N_t \approx \prod_{t=1}^{H-1}(n_t + 1)$	• N depends on M^t and on the number of new decision variables at each stage $N \approx \sum_{t=1}^{H-1} n_t \times M^t$
• A different adjustable variable x_t^i is constructed for any sample $\boldsymbol{\xi}^{t-1^{(i)}}$	• No adjustable variable

The robust multistage wait-and-see problem

If we **relax the non-anticipativity constraints** we have:

$$
\begin{aligned}
\mathrm{RWS}_H := \sup_{\boldsymbol{\xi}^{H-1}} \min_{x^1(\boldsymbol{\xi}^{H-1}),\dots,x^H(\boldsymbol{\xi}^{H-1})} \ & {c^1}^\top x^1(\boldsymbol{\xi}^{H-1}) + \ldots + {c^H}^\top x^H(\boldsymbol{\xi}^{H-1}) \\
\text{s.t.} \quad & Ax^1 = h^1, \ x^1 \geq 0 \\
& T^1(\xi^1)x^1(\boldsymbol{\xi}^{H-1}) + W^2(\xi^1)x^2(\boldsymbol{\xi}^{H-1}) = h^2(\xi^1), \\
& \vdots \\
& T^{H-1}(\boldsymbol{\xi}^{H-1})x^{H-1}(\boldsymbol{\xi}^{H-1}) + W^H(\boldsymbol{\xi}^{H-1})x^H(\boldsymbol{\xi}^{H-1}) = h^H(\boldsymbol{\xi}^{H-1}) \\
& x^t(\boldsymbol{\xi}^{H-1}) \geq 0 \ , \ t = 2, \ldots, H.
\end{aligned}
$$

Definition (The Robust Value of Perfect Information)

$$\mathrm{RVPI}_H := \mathrm{RO}_H - \mathrm{RWS}_H \geq 0$$

The robust two-stage relaxation

$$\mathrm{RT}_H \quad := \quad \min_{x^1} c^{1\top} x^1 + \sup_{\xi^1} \left[\min_{x^2,\dots,x^H} c^{2\top} x^2\left(\xi^1\right) + c^{3\top} x^3(\tilde{\boldsymbol{\xi}}^2) + \dots + c^{H\top} x^H(\tilde{\boldsymbol{\xi}}^{H-1}) \right]$$

$$\text{s.t. } Ax^1 = h^1,\ x^1 \geq 0$$

$$T^1(\xi^1)x^1 + W^2(\xi^1)x^2(\xi^1) = h^2(\xi^1),\ \forall \xi^1 \in \Xi^1$$

$$\vdots$$

$$T^{H-1}(\tilde{\boldsymbol{\xi}}^{H-1})x^{H-1}(\tilde{\boldsymbol{\xi}}^{H-1}) + W^H(\tilde{\boldsymbol{\xi}}^{H-1})x^H(\tilde{\boldsymbol{\xi}}^{H-1}) = h^H(\tilde{\boldsymbol{\xi}}^{H-1}),\ \forall \xi^1 \in \Xi^1$$

$$x^t(\tilde{\boldsymbol{\xi}}^{t-1}) \geq 0,\ t = 2,\dots,H,\ \forall \xi^1 \in \Xi^1.$$

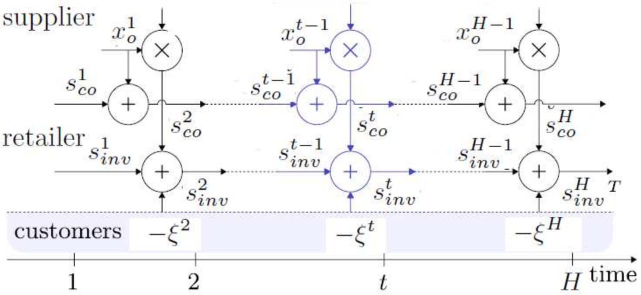
where $\tilde{\boldsymbol{\xi}}^t := (\xi^1, \tilde{\xi}^2, \dots, \tilde{\xi}^t), \quad t = 2, \dots, H-1$, with $\tilde{\xi}^t$ a deterministic realization of the random process ξ^t.

Proposition

$$RWS_H \leq RT_H \leq RO_H \quad \textit{lower bounds}$$

$$SWS_H^N \leq SRT_H^N \leq SRO_H^{N_1,N_2,\dots,N_{H-1}} \leq RO_H \quad \textit{sample-based bounds}$$

Case Study: Inventory Management with Cumulative Orders



[Vayanos, Kuhn, Rustem, Automatica (2012)]

Case Study: Inventory Management with Cumulative Orders

$$\mathrm{RO}_H(\mathcal{COC}) := \min_{x_o^t, x_c^t, s_{co}^t, s_{inv}^t} \left[x_c^1 + \max_{\boldsymbol{\xi} \in \Xi} \sum_{t \in \mathbb{T}} x_c^{t+1}(\boldsymbol{\xi}^t) \right] \quad \text{(1a)}$$

$$\text{s.t. } x_c^1 \geq d^1 x_o^1 + \max\left\{ h^1 s_{inv}^1, -p^1 s_{inv}^1 \right\} \quad \text{(1b)}$$

$$x_c^{t+1}(\boldsymbol{\xi}^t) \geq d^{t+1} x_o^{t+1}(\boldsymbol{\xi}^t) + + \max\left\{ h^{t+1} s_{inv}^{t+1}(\boldsymbol{\xi}^t), -p^{t+1} s_{inv}^{t+1}(\boldsymbol{\xi}^t) \right\}, \ t = 1, \ldots, H-2 \quad \text{(1c)}$$

$$x_c^H(\boldsymbol{\xi}^{H-1}) \geq \max\left\{ h^H s_{inv}^H(\boldsymbol{\xi}^{H-1}), -p^H s_{inv}^H(\boldsymbol{\xi}^{H-1}) \right\} \quad \text{(1d)}$$

$$s_{inv}^2(\boldsymbol{\xi}^1) = s_{inv}^1 + x_o^1 - \xi^1 \quad \text{(1e)}$$

$$s_{inv}^{t+1}(\boldsymbol{\xi}^t) = s_{inv}^t(\boldsymbol{\xi}^{t-1}) + x_o^t(\boldsymbol{\xi}^{t-1}) - \xi^t, \quad t = 2, \ldots, H-1 \quad \text{(1f)}$$

$$s_{co}^2(\boldsymbol{\xi}^1) = s_{co}^1 + x_o^1 \quad \text{(1g)}$$

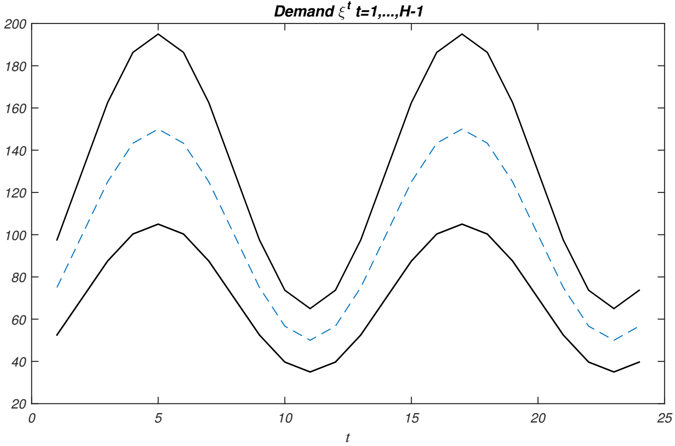
$$s_{co}^{t+1}(\boldsymbol{\xi}^t) = s_{co}^t(\boldsymbol{\xi}^{t-1}) + x_o^t(\boldsymbol{\xi}^{t-1}), \quad t = 2, \ldots, H-1 \quad \text{(1h)}$$

$$\underline{x}_o^1 \leq x_o^1 \leq \bar{x}_o^1, \quad \underline{s}_{co}^1 \leq s_{co}^1 \leq \bar{s}_{co}^1 \quad \text{(1i)}$$

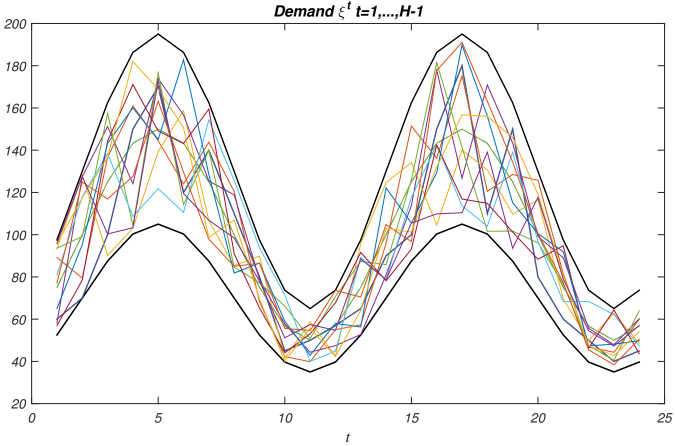
$$\underline{x}_o^t \leq x_o^t(\boldsymbol{\xi}^{t-1}) \leq \bar{x}_o^t, \quad \underline{s}_{co}^t \leq s_{co}^t(\boldsymbol{\xi}^{t-1}) \leq \bar{s}_{co}^t, \quad t = 2, \ldots, H. \quad \text{(1j)}$$

[Vayanos, Kuhn, Rustem, Automatica (2012)]

Uncertain Demand



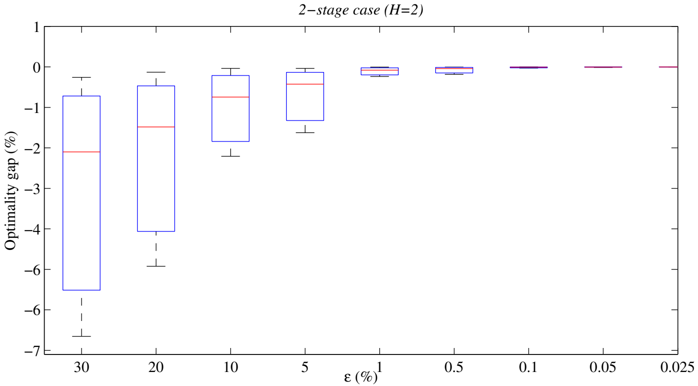
Scenarios for Uncertain Demand



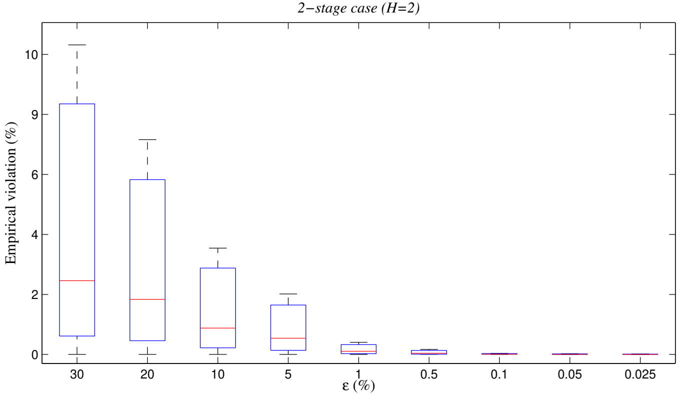
Optimality Gaps

We define **optimality gaps** of the problem $\mathrm{SwC}_2^N(\mathcal{COC})$ as

$$optimality\ gap := \frac{\inf \mathrm{SwC}_2^N(\mathcal{COC}) - \inf \mathrm{RO}_2(\mathcal{COC})}{\inf \mathrm{RO}_2(\mathcal{COC})}$$



Empirical Violation Probability



- As ϵ decreases, the **violation converges to 0**.
- The **empirical violation probability is smaller than** ϵ in all the cases.

Number of samples

ϵ (%)	N	# of const. $(H = 2)$	# of var. $(H = 2)$	ϵ_t (%)	$N = N_1 N_2 N_3 N_4$
30	63	756	442	-	-
20	95	1140	666	20	78704572
10	189	2268	1324	10	1259273144
5	377	4524	2640	5	20148370300
1	1884	22608	13189	1	12592731437183
0.5	3768	45216	26377	0.5	201483702994927
0.1	18838	226056	131867	0.1	125927314371829000
0.05	37676	452112	263733	0.05	201483702994926000
0.025	75352	904224	527465	0.025	3223739247918820000

- Five-stage case: **out-of-memory**[1] due to the large number of required samples N.
- We consider **relaxations** for problem $\mathrm{RO}_H(\mathcal{COC})$:
  1. **multistage wait-and-see problem** $\mathrm{RWS}_H(\mathcal{COC})$;
  2. **Robust two-stage relaxation problem** $\mathrm{RT}_H(\mathcal{COC})$ (where the nonanticipativity constraints are relaxed in stages $2, \ldots, H$);
  3. **Sampled two-stage relaxation problem** $\mathrm{SRT}_H(\mathcal{COC})$.

[1] 64-bit machine with 12 GB of RAM and a Intel Core i7-3520M CPU 2.90 GHz processor.

Lower Bounds

- **Robust Value of Perfect Information**

$$RVPI_5(\mathcal{COC}) = RO_5(\mathcal{COC}) - RWS_5(\mathcal{COC}) = 2207.55 - 1831.89 = 375.66$$

- **Lower Bounds**

$$\mathrm{RWS}_H \leq \mathrm{RT}_H \leq \mathrm{RO}_H.$$

$$(optimality\ gap)_{\mathrm{RWS}_5(\mathcal{COC})} := \frac{\mathrm{RWS}_5(\mathcal{COC}) - \mathrm{RO}_5(\mathcal{COC})}{\mathrm{RO}_5(\mathcal{COC})} = -17\%$$

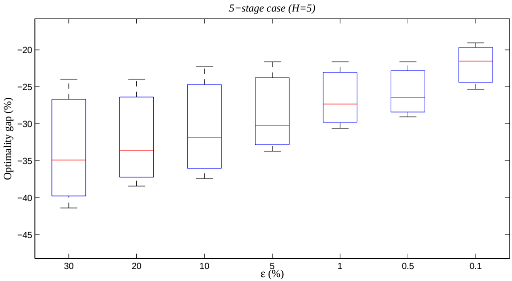
$$(optimality\ gap)_{\mathrm{RT}_5(\mathcal{COC})} := \frac{\mathrm{RT}_5(\mathcal{COC}) - \mathrm{RO}_5(\mathcal{COC})}{\mathrm{RO}_5(\mathcal{COC})} = -17\%$$

- **Sample-based Lower Bounds**

$$\mathrm{SWS}_H^N \leq \mathrm{SRT}_H^N \leq \mathrm{SRO}_H^{N_1, N_2, \ldots, N_{H-1}} \leq \mathrm{RO}_H.$$

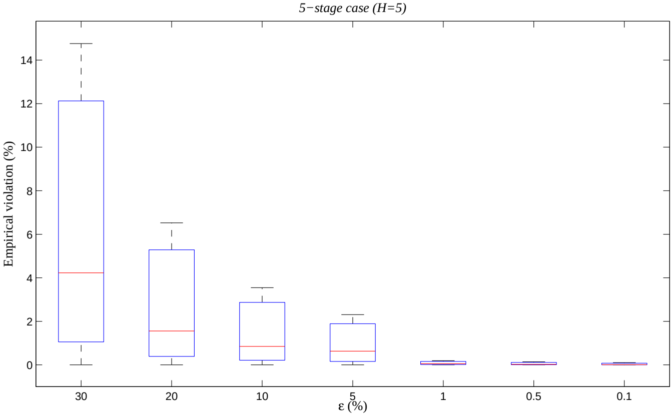
Optimality Gaps: 5-stage case

$$optimality\ gap := \frac{\inf \mathrm{SRT}_H^N(\mathcal{COC}) - \inf \mathrm{RO}_H(\mathcal{COC})}{\inf \mathrm{RO}_H(\mathcal{COC})}$$



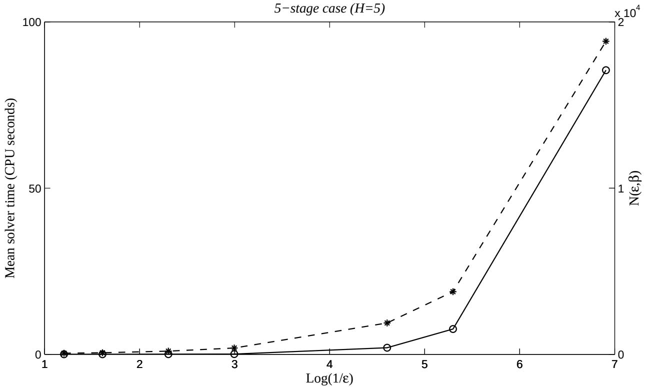
- For a given ϵ, the $\mathrm{SRT}_5^N(\mathcal{COC})$ cost is **lower** than the solution returned by **decision rules approximations**.

Empirical Violation Probability: 5-stage case



- As ϵ decreases, the **violation converges to 0**;
- The **empirical violation probability is smaller than** ϵ in all the cases.

Mean solver times and number of samples: 5-stage case



- The **number of samples**:
  - does **not depend** on the **number of stages**;
  - is the **same** of the **two stage** case;
  - is considerably **lower than those used in literature** where it depends on the size of the basis and on the number of decision variables at each stage.

Conclusions and References

- **Probabilistic guarantees** for **constraint sampling** in **multistage convex robust optimization** problems have been proposed.
- **Bound on the probability of violation** for the scenario approach has been provided.
- **Bounds** have been provided also by relaxing the nonanticipativity constraints.
- **Advantage** of the proposed approach: avoid the conservative use of parametrization through decision rules.

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1. Calafiore, G., Campi, M.: **Uncertain convex programs: randomized solutions and confidence levels**, Mathematical Programming **102**(1), 25–46 (2004).
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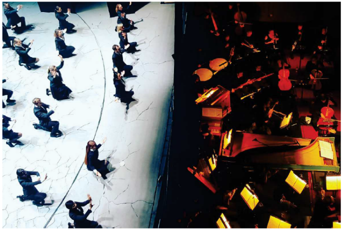
Current Research

- Bomze, I. and Maggioni, F: **Two-stage Stochastic Optimization of StQPs for Clustering in Social Networks**, in preparation.



Happy 60th Birthday!


Happy Birthday
Manuel
60