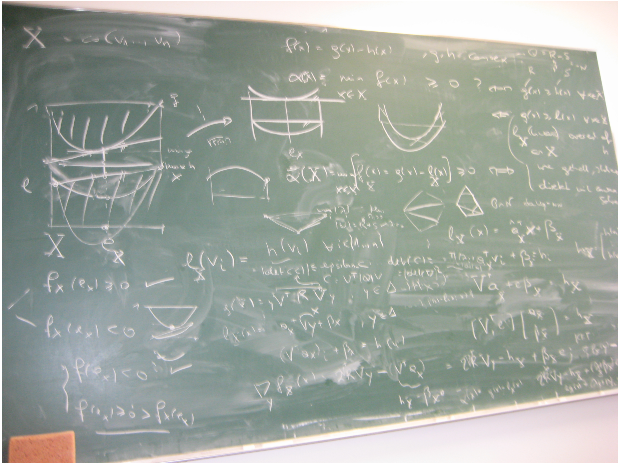
Application of Convex Underestimators to Global Multiobjective Optimization

Gabriele Eichfelder
joint work with Tobias Gerlach, Julia Niebling, and Susanne Sumi (Ilmenau)

How it started . . .



(but also other great memories of Vienna!)

Copositivity test...

Matrix $A \in \mathcal{S}^n$ copositive $\Leftrightarrow$ $\min_{x \geq 0} x^\top A x = 0$

Copositivity test. . .

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- Copositivity detection via difference-of-convex approach: Decompose $Q = Q_+ - Q_-$ with Q_+, Q_- positive semidefinite, and replace by

$$\inf\{x^\top Q_+ x \mid x^\top Q_- x = 1,\ x \geq 0\} \geq 1.$$

and . . . $x^\top Q x = x^\top Q_+ x - x^\top Q_- x \geq x^\top Q_+ x - 1 \geq 0$

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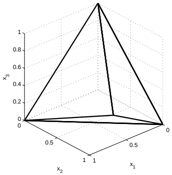
- Branch-and-Bound by partitioning of the standard simplex:



Content

1. αBB method for global (scalar) optimization
   $f : \mathbb{R}^n \to \mathbb{R}$

2. Basics of multiobjective optimization
   $f : \mathbb{R}^n \to \mathbb{R}^m$

3. αBB method for global multiobjective optimization
   $f : \mathbb{R}^n \to \mathbb{R}^m$

Nonlinear Optimization

We consider the following nonlinear optimization problem

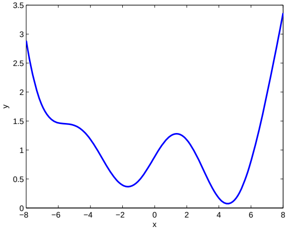
$$\min_{x \in X} f(x) \tag{P}$$

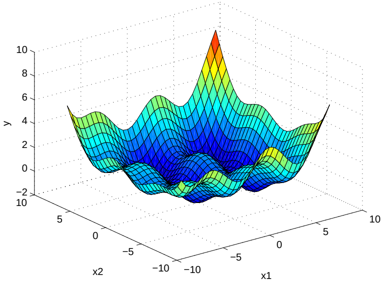
with twice continuously differentiable objective function $f : \mathbb{R}^n \to \mathbb{R}$ and feasible set

$$X = \{x \in \mathbb{R}^n \mid \underline{x} \leq x \leq \overline{x}\} = [\underline{x}, \overline{x}]$$

with $\underline{x}, \overline{x} \in \mathbb{R}^n$, $\underline{x} \leq \overline{x}$.

Local and Global Solutions of $\min_{x \in X} f(x)$ (P)





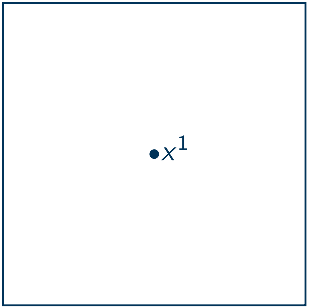
- A point $\bar{x} \in X$ is a (global) minimal solution of (P), if

$$f(\bar{x}) \leq f(x) \ \text{ for all } \ x \in X.$$

- A point $\bar{x} \in X$ is a local minimal solution of (P), if a neighbourhood $U_{\bar{x}}$ of $\bar{x}$ exists with

$$f(\bar{x}) \leq f(x) \ \text{ for all } \ x \in X \cap U_{\bar{x}}.$$

Basic Branch and Bound Algorithm

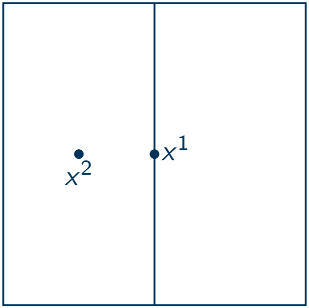


Feasible set $X \subset \mathbb{R}^2$

- Branch: Partitioning rule, partition in subboxes X^i
- Selection rule (e.g. $f(x^i)$ with midpoints x^i of subboxes X^i)
- Bound: Discarding tests
- Termination rule

Basic idea: $\min_{x \in X} f(x) = \min\{\min_{x \in X^i} f(x)\}$ if $X = \bigcup_i X^i$

Basic Branch and Bound Algorithm



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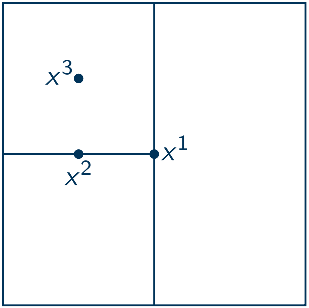

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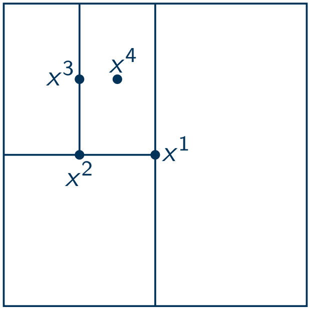

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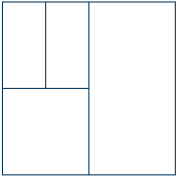
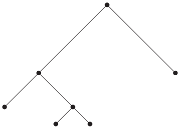
Discarding tests scalar-valued case $f : X \to \mathbb{R}$





- upper bound for the (global) minimum by $\min_i f(x^i)$ (with midpoints x^i)
- lower bounds by interval arithmetic or convex underestimators for the values $f(x)$ with $x \in X^j$ for each subbox X^j, i.e. ω^j with $\omega^j \leq \min_{x \in X^j} f(x)$
- Discard subbox X^j if

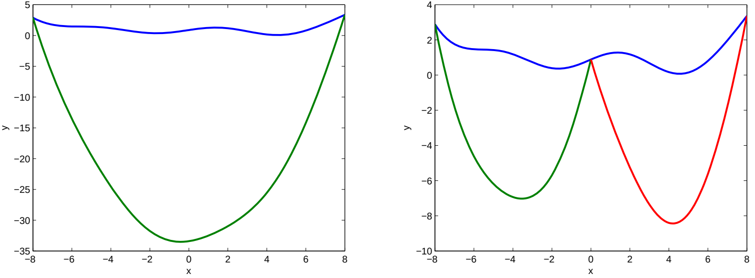
$$\min_i f(x^i) < \omega^j.$$

αBB: Bounds by Convex Underestimator

Definition

As before let $X \subset \mathbb{R}^n$ and $f\colon \mathbb{R}^n \to \mathbb{R}$. A function $\tilde{f}\colon X \to \mathbb{R}$ is called a **convex underestimator** of f on X, if

- $\tilde{f}$ is convex on X and
- $\tilde{f}(x) \leq f(x)$ for all $x \in X$.



Convex Underestimator by αBB

Lemma (Maranas, Floudas '94)

As before let $X = [\underline{x}, \overline{x}]$*. Then* $f_{\alpha,X} \colon X \to \mathbb{R}$*,*

$$f_{\alpha,X}(x) := f(x) + \alpha \sum_{i=1}^{n} \underbrace{(\underline{x}_i - x_i)(\overline{x}_i - x_i)}_{\leq 0},$$

is a convex underestimator of f *w.r.t.* X *if and only if*

$$\alpha \geq \max\left\{0,\ -\frac{1}{2}\min\left\{\lambda_{\min}(H_f(x)) \mid x \in X\right\}\right\}.$$

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It holds:

$$u_j := \min_{x \in X^j} f_{\alpha,X^j}(x) \leq \min_{x \in X^j} f(x)$$

Convex Underestimator by αBB

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It holds:

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and

$$\max_{x \in X}\left(f(x) - f_{\alpha,X}(x)\right) = \frac{\alpha}{8}\|\overline{x} - \underline{x}\|_2^2.$$

Algorithm for (ε, δ)-minimal set

Aim: Cover/representation of the set $\operatorname{argmin}\{f(x) \mid x \in X\}$.

Definition

Let $\varepsilon > 0$, $\delta > 0$ and $\operatorname{argmin}_{x \in X} f(x) \neq \emptyset$.

(a) $\tilde{x} \in X$ is an ε-*minimal point* of f w.r.t. X, if

$$f(\tilde{x}) \leq \min_{x \in X} f(x) + \varepsilon.$$

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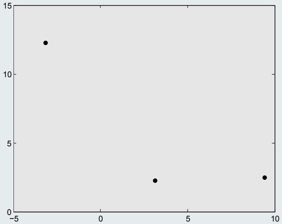
$$f(\tilde{x}) \leq \min_{x \in X} f(x) + \varepsilon.$$

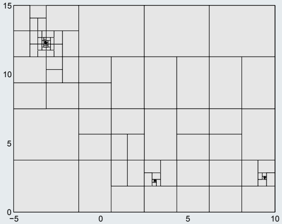
(b) A finite subset A of ε-minimal points of f w.r.t. X is an (ε, δ)-*minimal set* of f w.r.t. X if for all $\bar{x} \in \operatorname{argmin}_{x \in X} f(x)$ there exists $\hat{x} \in A$ such that $\|\hat{x} - \bar{x}\|_2 \leq \delta$.

Modified αBB Algorithm

Beispiel (Branin, Hoo '72)

$f(x) := \left(x_2 - \frac{5{,}1}{4\pi^2}x_1^2 + \frac{5}{\pi}x_1 - 6\right)^2 + 10\left(1 - \frac{1}{8\pi}\right)\cos(x_1) + 10$
$X = [-5, 10] \times [0, 15]$.

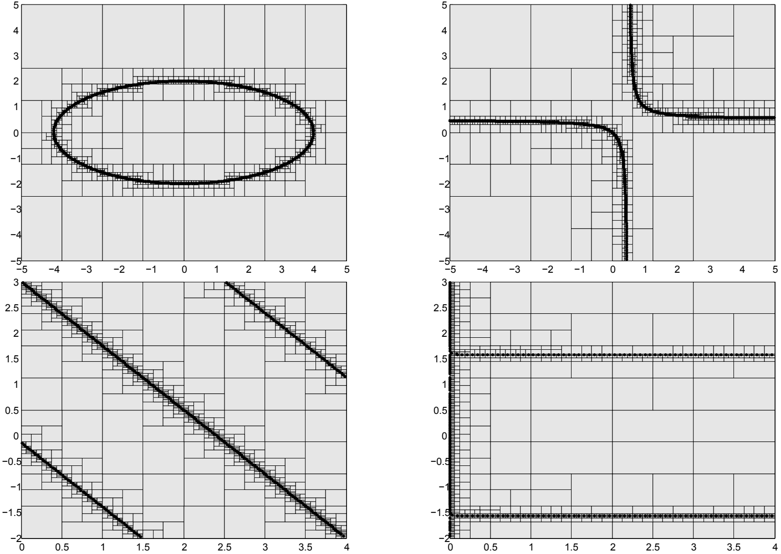




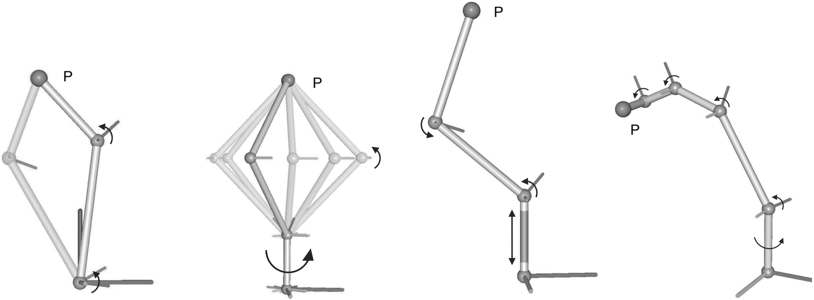
Theorem (E./Gerlach/Sumi '16)

The modified αBB algorithm terminates after finitely many iterations with an (ε, δ)-minimal set of f w.r.t. X.

Some examples



Robot arm (Tetra GmbH, Ilmenau)



2

Basics of multiobjective optimization

Multiobjective Optimization

$$\min \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix} \text{s.t. } x \in X \qquad \text{(MOP)}$$

with twice continuously differentiable functions
$f_i \colon \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$,
and feasible set

$$X = \{x \in \mathbb{R}^n \mid \underline{x} \leq x \leq \overline{x}\} = [\underline{x}, \overline{x}]$$

where $\underline{x}, \overline{x} \in \mathbb{R}^n$ with $\underline{x} \leq \overline{x}$.

Efficient solutions in multi-objective optimization

Consider

$$\min_{x \in X} f(x) \quad \text{with} \quad f \colon \mathbb{R}^n \to \mathbb{R}^m,\ f(x) = (f_1(x), \ldots, f_m(x))^\top.$$

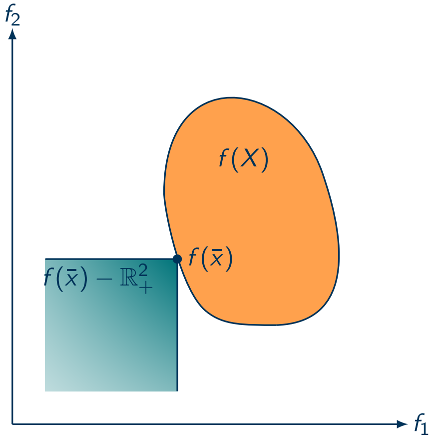
The point $\bar{x} \in X$ is an **efficient solution** of $\min_{x \in X} f(x)$ if there exists **no** $x \in X$ with

$$\begin{array}{rccl} & f_i(x) & \leq & f_i(\bar{x}) \ \text{ for all } i = 1, \ldots, m, \\ \text{and} & f_j(x) & < & f_j(\bar{x}) \ \text{ for at least one } j \in \{1, \ldots, m\}. \end{array}$$

Hence if

$$(\{f(\bar{x})\} - \mathbb{R}^m_+) \cap f(X) = \{f(\bar{x})\} \ .$$

Efficient solutions with $m = 2$



$$\bar{x} \text{ is efficient} \iff (\{f(\bar{x})\} - \mathbb{R}^2_+) \cap f(X) = \{f(\bar{x})\}$$

Scalarization

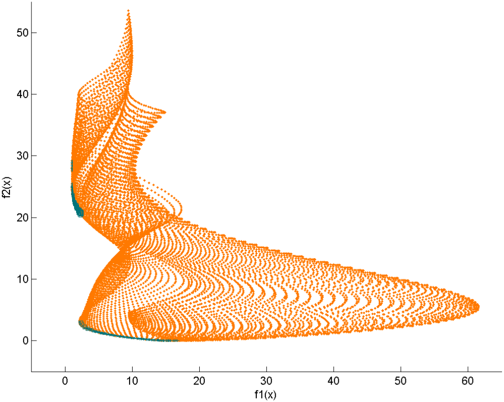
- Weighted sum

$$\min_{x \in X} w^\top f(x) = \sum_{i=1}^{m} w_i f_i(x) = w_1 f_1(x) + \ldots + w_m f_m(x).$$

- ε-constraint method

$$\begin{gathered} \min \; f_m(x) \\ \text{s.t.} \\ f_1(x) \leq \varepsilon_1 \\ \vdots \\ f_{m-1}(x) \leq \varepsilon_{m-1} \\ x \in X \end{gathered}$$

Local efficient solutions: Poloni function



3

αBB method for global multiobjective optimization

Basic Branch-and-Bound for biobjective problems

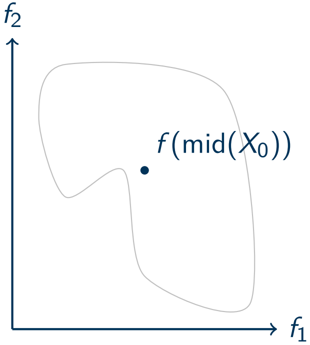
$$\min \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \text{s.t. } x \in X_0 = [\underline{x}, \overline{x}] \qquad \text{(MOP)}$$

- Branch: Partitioning rule
- Selection rule
- Bound: Discarding tests
  - upper bound?
  - lower bound on subboxes?
- Termination rule

[Fernández, Tóth '09]

Multiobjetive optimization: upper bounds?

„Provisional nondominated solutions“ $\mathcal{L}_{\mathrm{PNS}}$:
List with known objective values, which do not dominate each other.



wobei $\mathrm{mid}(X_0) := \frac{1}{2}(\overline{x} + \underline{x})$ für $X_0 = [\underline{x}, \overline{x}]$.

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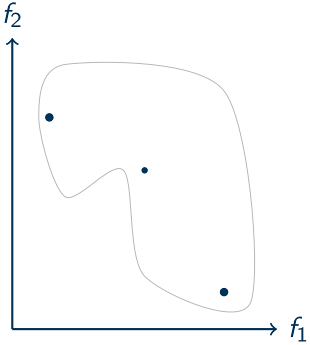

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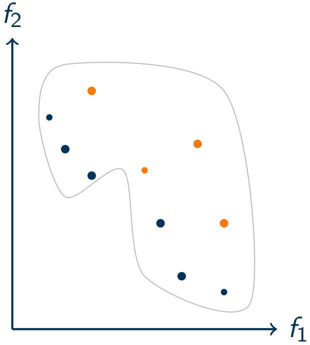

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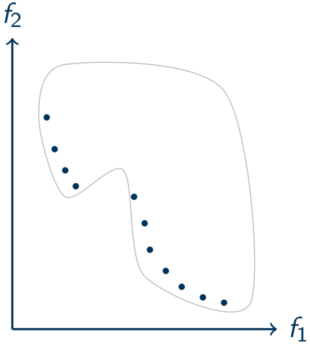

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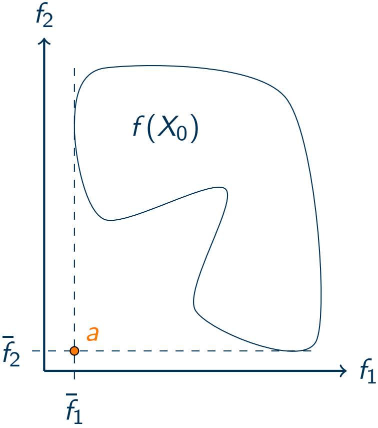
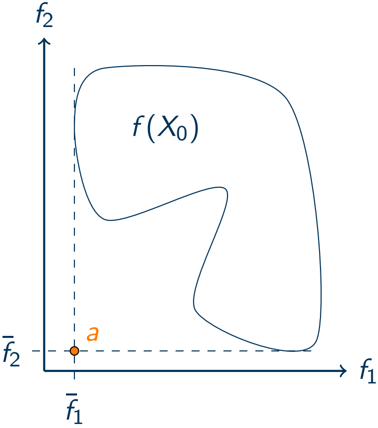
Lower bounds? Ideal point



- Solve
  $\bar{f}_1 := \min_{x \in X_0} f_1(x)$ and
  $\bar{f}_2 := \min_{x \in X_0} f_2(x)$
- Set $a = (\bar{f}_1, \bar{f}_2)^\top$

Lower bounds? Ideal point



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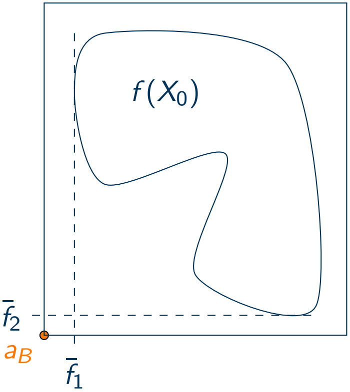
Ideal point and interval arithmetic

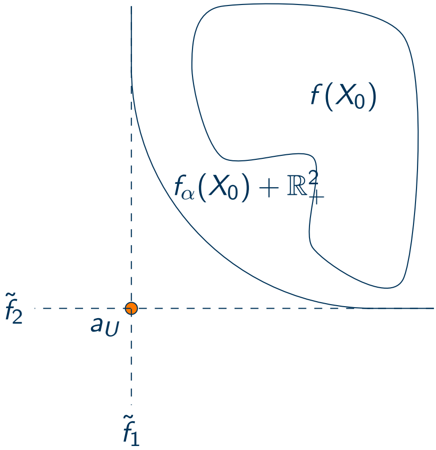


- Determine interval extensions F_1 of f_1 and F_2 of f_2 and use that for calculating a box $B \subset \mathbb{R}^2$ with

$$f(X_0) \subset B$$

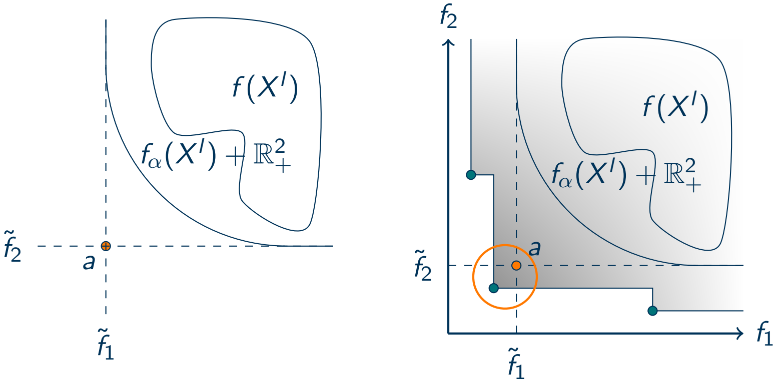
- Choose a_B als lower bound of the box B

Ideal point and convex underestimators



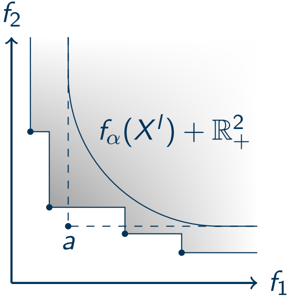
- Determine convex underestimators $f_{1,\alpha}$ of f_1 and $f_{2,\alpha}$ of f_2 on X_0
- Calculate $\tilde{f}_1 := \min_{x \in X_0} f_{1,\alpha}(x)$ and $\tilde{f}_2 := \min_{x \in X_0} f_{2,\alpha}(x)$
- Set $a_U := (\tilde{f}_1, \tilde{f}_2)^\top$

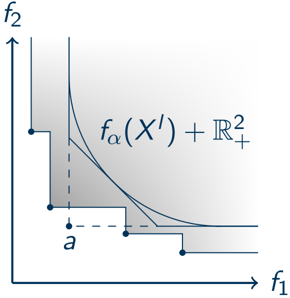
Fist discarding test for a box X^l



Compare a with points from the list $\mathcal{L}_{\mathrm{PNS}}$(=upper bounds) and discard X^l, if a is componentwise larger than any point from $\mathcal{L}_{\mathrm{PNS}}$!

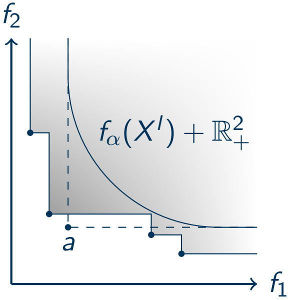
Improved discarding test



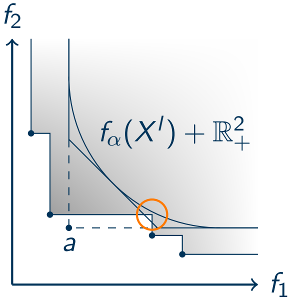


Improve lower bound by applying one step of **Bensons algorithm** [Benson '98, Ehrgott, Shao, Schöbel '11, Löhne, Rudloff, Ulus '14] with objective function f_α in X^I.

Further Improvement



ideal point with
convex underestimator



Benson step

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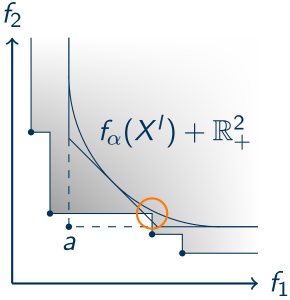

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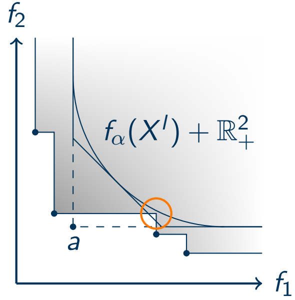

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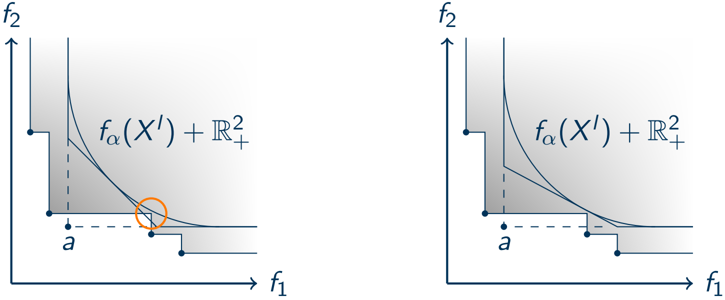
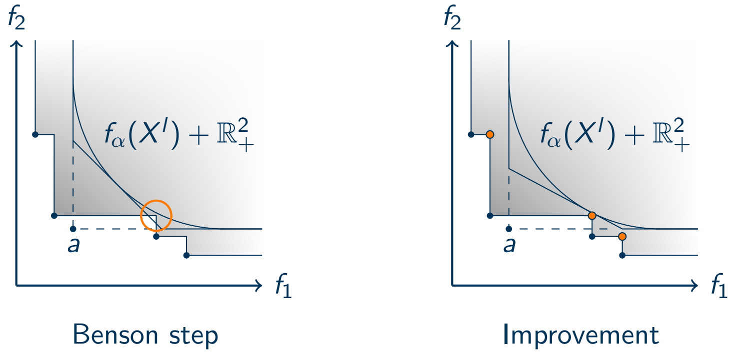


Benson step

Improvement

Further Improvement



Apply Bensons algorithm at *local upper bounds*.

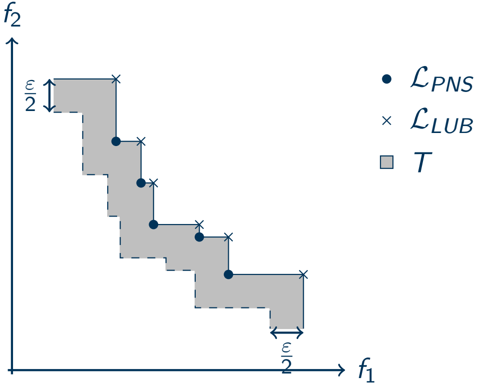
Interval arithmetic vs. Benson improved

n-dimensional *Fonseca Fleming* function:

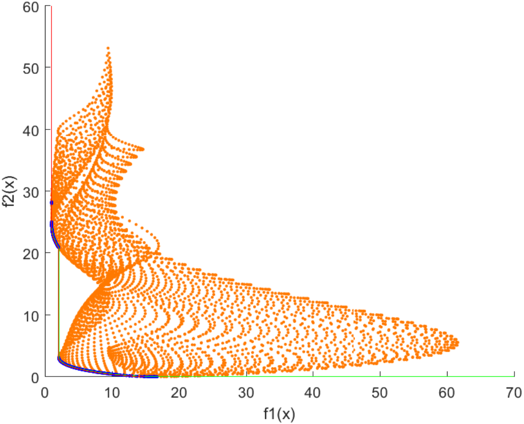
$$f(x) = \begin{pmatrix} 1 - \exp\left(-\sum_{i=1}^{n}\left(x_i - \frac{1}{\sqrt{n}}\right)^2\right) \\ 1 - \exp\left(-\sum_{i=1}^{n}\left(x_i + \frac{1}{\sqrt{n}}\right)^2\right) \end{pmatrix}, \; X_0 = \left[\begin{pmatrix}-2\\ \vdots \\ -2\end{pmatrix}, \begin{pmatrix}2\\ \vdots \\ 2\end{pmatrix}\right] \subset \mathbb{R}^n$$

	Interval arithmetic			Benson improved		
n	$\lvert\mathcal{L}_S\rvert$	iterations	time [s]	$\lvert\mathcal{L}_S\rvert$	iterations	time [s]
2	258	373	74.12	177	**273**	84.65
3	2108	3563	664.98	810	**1795**	634.98
4	17934	33394	6.71e+3	3925	**11466**	4.81e+3
5	199460	325595	9.91e+4	19622	**53221**	2.94e+4

Some theoretical results



Poloni function



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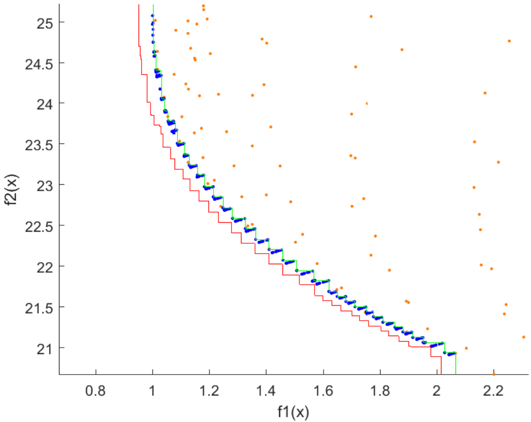

Some theoretical results

- Let for some predefined $\varepsilon > 0$

$$T := \left(\bigcup_{\bar{p} \in \mathcal{L}_{LUB}} \{\bar{p}\} - \mathbb{R}^m_+ \right) \setminus \left(\bigcup_{\bar{p} \in \mathcal{L}_{LUB}} \left\{ \bar{p} - \frac{\varepsilon}{2} e \right\} - \operatorname{int}(\mathbb{R}^m_+) \right).$$

Then $\mathcal{L}_{PNS} \subseteq T$ and for any efficient point $\bar{x}$ it holds $f(\bar{x}) \in T$.

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  Then $\mathcal{L}_{PNS} \subseteq T$ and for any efficient point $\bar{x}$ it holds $f(\bar{x}) \in T$.
- Let $\mathcal{A}$ be the set generated by the algorithm. Then any $\tilde{x} \in \mathcal{A}$ is an ε-minimal point. And for any efficient point y there is a point $\tilde{x} \in \mathcal{A}$ with $\|\tilde{x} - y\| < \delta$.

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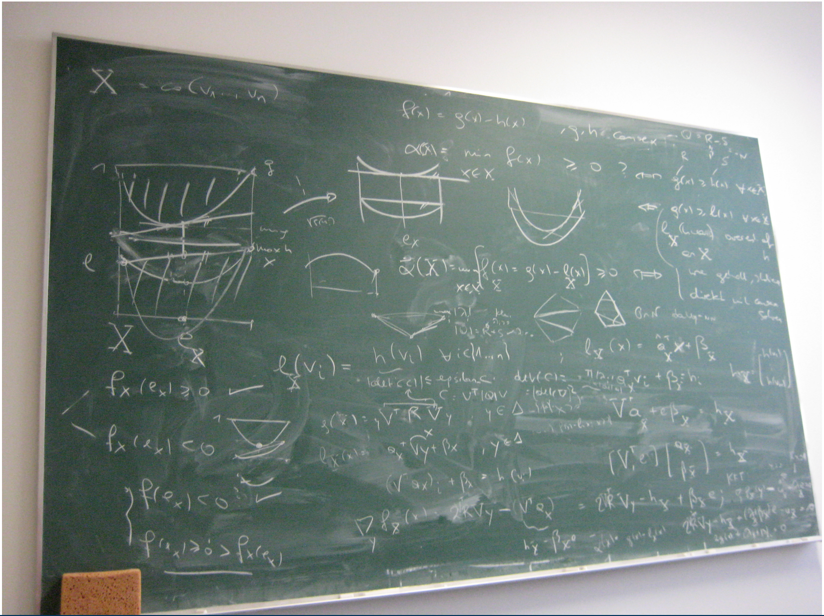
- All while loops in the algorithm are finite.



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