Copositive optimization and completely positive matrix factorization

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Joint work with Patrick Groetzner

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First of all ...

Congratulations to

Manuel

and many happy returns!

Standard Quadratic Problem

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Theorem (Bomze, D., et al. 2000)

StQP is equivalent to the following linear problem over $\mathcal{CP}$:

$$\min \langle A, X \rangle \text{ s. t. } \langle E, X \rangle = 1, \; X \in \mathcal{CP}.$$

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More precisely:

If x^* is optimal for StQP $\implies$ $x^*(x^*)^T$ is optimal for cpStQP

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So: given $X \in \mathcal{CP}$, how to find factorization $X = BB^T$?

Factorizations are not unique

Example (Dickinson 2010)

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The minimal possible number of columns is called the cp-rank of A:

$$\operatorname{cprank}(A) = \inf\{r \in \mathbb{N} \mid \exists B \in \mathbb{R}^{n\times r}, B \geq 0, A = BB^T\}$$

Orthogonal matrices

Good news:

$\exists$ upper bound cp_n such that

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Lemma (Bomze, Dickinson, Still 2015): For all $A \in \mathcal{CP}_n$ we have:

$$\mathrm{cprank}(A) \leq \mathrm{cp}_n = \begin{cases} n+1 & n \in \{2,3,4\} \\ \frac{1}{2}n(n+1)-3 & n \geq 5 \end{cases}$$

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Fundamental Lemma for our approach:

Let $B, C \in \mathbb{R}^{n\times r}$. Then $BB^T = CC^T$ if and only if there exists an orthogonal matrix $Q \in \mathbb{R}^{r\times r}$ with $BQ = C$.

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- Find $Q \in \mathcal{O}_r$ with $BQ \geq 0$
- Output: $A = (BQ)(BQ)^T$ is a $\mathcal{CP}$ factorization

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Feasibility Problem:

$$\begin{aligned} &\text{find } Q \\ &\text{s.t. } BQ \geq 0 \\ &\qquad Q \in \mathbb{R}^{r \times r} \text{ orthogonal} \end{aligned}$$

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Consider polyhedral cone $\mathcal{P} := \{Q \in \mathbb{R}^{r\times r} \mid BQ \geq 0\}$

and the manifold $\mathcal{O}_r := \{Q \in \mathbb{R}^{r\times r} \mid QQ^T = I_{r\times r}\}$

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$\Longrightarrow$ use alternating projection method between $\mathcal{P}$ and $\mathcal{O}_r$.

Example

Given A, the eigen-decomposition gives an initial factorization

$$\underbrace{\begin{pmatrix} 4 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 0.0000 & 0.0000 & 0.0000 & -1.2308 & 1.5764 \\ 0.5413 & -0.5152 & 0.4377 & 0.4805 & 1.0950 \\ 0.0533 & 0.8062 & 0.3119 & 0.4805 & 1.0950 \\ -0.8108 & -0.2910 & -0.0890 & 0.4805 & 1.0950 \\ 0.2163 & 0.0000 & -0.8386 & 0.4805 & 1.0950 \end{pmatrix}}_{\tilde{B}} \cdot \underbrace{\begin{pmatrix} 0.0000 & 0.0000 & 0.0000 & -1.2308 & 1.5764 \\ 0.5413 & -0.5152 & 0.4377 & 0.4805 & 1.0950 \\ 0.0533 & 0.8062 & 0.3119 & 0.4805 & 1.0950 \\ -0.8108 & -0.2910 & -0.0890 & 0.4805 & 1.0950 \\ 0.2163 & 0.0000 & -0.8386 & 0.4805 & 1.0950 \end{pmatrix}^T}_{\tilde{B}^T}$$

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Alternative: Add zero-columns $\rightarrow$ numerically not so promising

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Solution after 83 iterations and 25 milliseconds:

$$BQ_k = \begin{pmatrix} 0.0265 & 0.0000 & 0.0621 & 0.0000 & 0.0000 & 0.1058 & 0.0011 & 0.6029 & 0.0000 & 0.8386 & 1.7081 & 0.0000 \\ 0.0239 & 0.0000 & 0.1370 & 0.0000 & 0.0000 & 0.1175 & 0.0001 & 0.0000 & 0.0067 & 1.0477 & 0.0584 & 0.9304 \\ 0.0000 & 0.0719 & 0.3288 & 0.0000 & 0.0010 & 0.1472 & 0.0016 & 0.0000 & 1.1392 & 0.4069 & 0.3646 & 0.5184 \\ 0.0194 & 0.0000 & 0.9413 & 0.0015 & 0.1146 & 0.2921 & 0.0000 & 0.0000 & 0.0062 & 0.0073 & 0.5292 & 0.8573 \\ 0.1175 & 0.0000 & 0.6165 & 0.0000 & 0.0000 & 0.9972 & 0.1690 & 0.0274 & 0.2500 & 0.7074 & 0.1437 & 0.0506 \end{pmatrix} \geq 0$$

Example

Now find $Q \in \mathcal{O}_{12}$ with $BQ \geq 0$.

$$Q_0 = I \quad \longrightarrow \quad \textbf{our algorithm} \quad \longrightarrow \quad BQ_k$$

$$\downarrow$$

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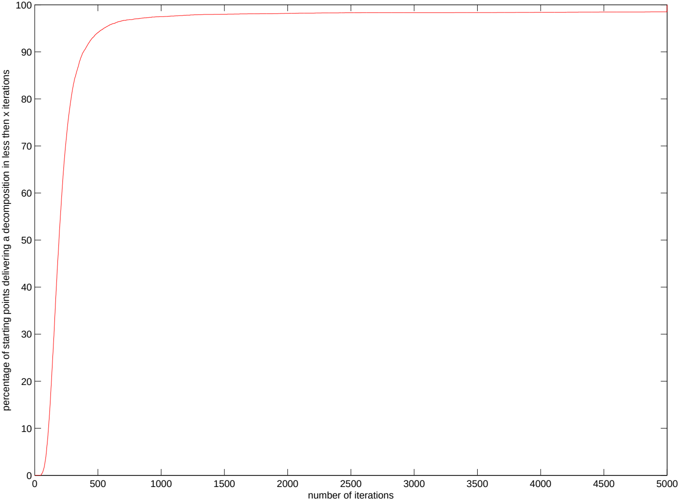
Hence: $A = (BQ_k)(BQ_k)^T \in \mathcal{CP}_5$!

Performance with 10,000 different Q_0



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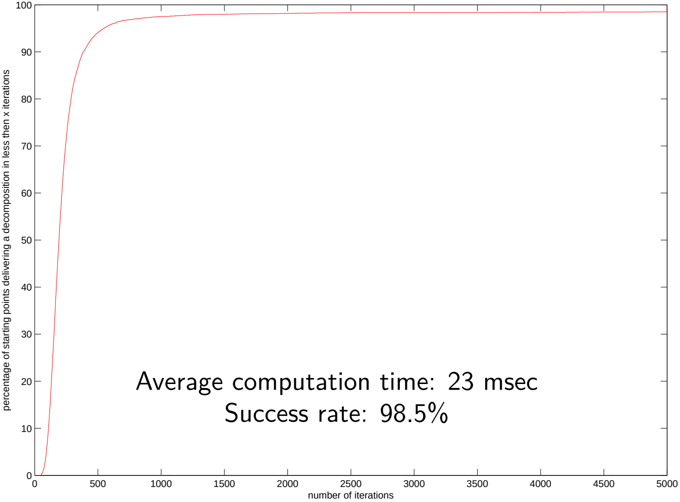
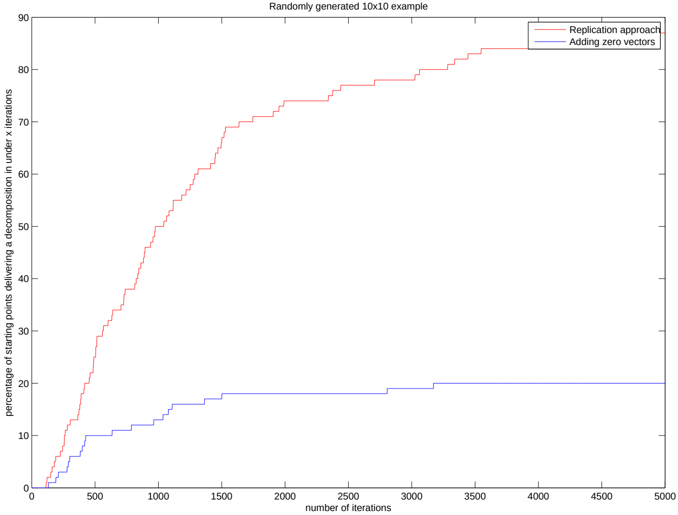


Why not add zero-columns?



Results on randomly generated instances

- construct $R \in \mathbb{R}^{n \times r}, R \geq 0$
- $A := RR^T$
- let $A = \tilde{B}\tilde{B}^T$ be the Choleski-Factorization of A
- construct B by column replication
- run our algorithm

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See preprint:

P.Groetzner and M.Dür: A factorization method for completely positive matrices. *Preprint* 2018, avilable on OptimizationOnline

Thank you for your attention!

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ALGORITHMIC OPTIMIZATION