Generating irreducible copositive matrices using the stable set problem

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Rabobank and University of Twente

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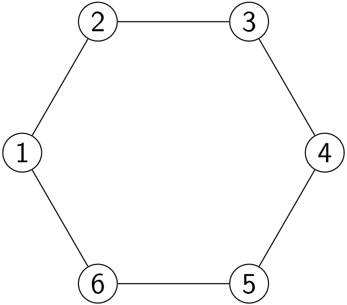

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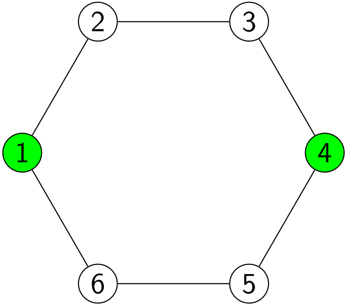

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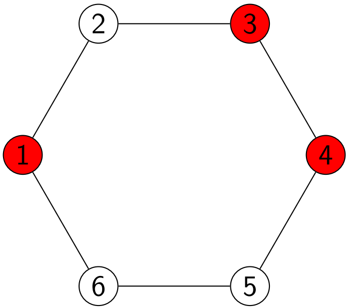

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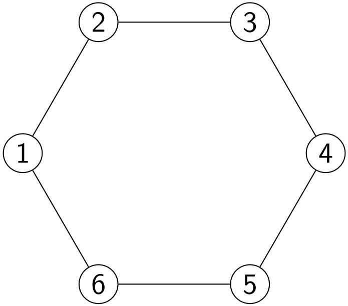

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$\alpha(G) := \max_{\mathcal{J}}\{|\mathcal{J}| : \mathcal{J}$ is a stable set of $G\}$.

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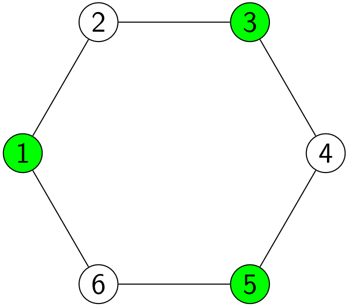

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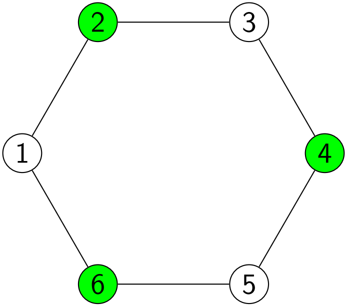

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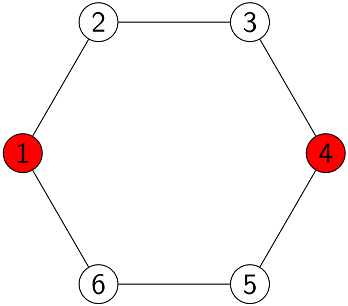

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Copositivity

Definition

Copositive Cone, $\mathcal{COP}^n := \{X \in \mathcal{S}^n \mid \mathbf{v}^\mathsf{T} X \mathbf{v} \geq 0 \text{ for all } \mathbf{v} \in \mathbb{R}^n_+\}$.

Theorem ([Murty and Kabadi, 1987],[D. and Gijben, 2013])

Checking copositivity is co-NP-complete.

Application of Copositivity to Stable Set Problem

Theorem

$$\alpha(G) = \min_{\lambda \in \mathbb{R}} \{\lambda : \lambda(\mathrm{I} + \mathsf{A}_G) - \mathsf{E} \in \mathcal{COP}^n\}.$$

[Motzkin and Straus, 1965] [Bomze, 1998] [Bomze et al., 2000] [de Klerk and Pasechnik, 2002] [D., 2013]

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Lemma

$\mathcal{COP}^n$ is proper cone with $\mathcal{PSD}^n + \mathcal{N}^n \subseteq \mathcal{COP}^n$.

$$(\underbrace{\mathrm{I}}_{\in \mathrm{int}(\mathcal{PSD}^n)} + \underbrace{\mathrm{A}_G}_{\in \mathcal{N}^n}) \in \mathrm{int}(\mathcal{PSD}^n + \mathcal{N}^n) \subseteq \mathrm{int}(\mathcal{COP}^n).$$

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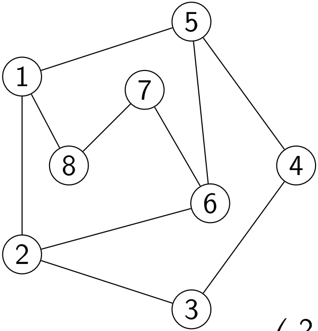
Lemma

$$\lambda(\mathrm{I} + \mathrm{A}_G) - \mathrm{E} \in \mathrm{bd}(\mathcal{COP}^n) \quad \Leftrightarrow \quad \lambda = \alpha(G)$$

Let $\mathrm{Z}_G := \alpha(G)\,(\mathrm{I} + \mathrm{A}_G) - \mathrm{E}$.

Example



$$A_G = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$n = 8,$$

$$\alpha(G) = 3,$$

$$Z_G = \begin{pmatrix} 2 & 2 & -1 & -1 & 2 & -1 & -1 & 2 \\ 2 & 2 & 2 & -1 & -1 & 2 & -1 & -1 \\ -1 & 2 & 2 & 2 & -1 & -1 & -1 & -1 \\ -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 \\ 2 & -1 & -1 & 2 & 2 & 2 & -1 & -1 \\ -1 & 2 & -1 & -1 & 2 & 2 & 2 & -1 \\ -1 & -1 & -1 & -1 & -1 & 2 & 2 & 2 \\ 2 & -1 & -1 & -1 & -1 & -1 & 2 & 2 \end{pmatrix}.$$

Irreducibility and Extremality

Lemma

$\mathcal{COP}^n$ is proper cone with $\mathcal{PSD}^n + \mathcal{N}^n \subseteq \mathcal{COP}^n$.

Definition

$X \in \mathcal{COP}^n$ is an extreme copositive matrix, $X \in \mathrm{Ext}(\mathcal{COP})$, if $\nexists A \in \mathcal{COP}^n \setminus (\mathbb{R}\{X\})$ such that $X - A \in \mathcal{COP}^n$.

$\mathrm{Ext}(\mathcal{COP}^n) \cap (\mathcal{PSD}^n + \mathcal{N}^n)$ known [Hall and Newman, 1963].

$\mathrm{Ext}(\mathcal{COP}^5)$ characterised in [Hildebrand, 2012].

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Definition

A matrix $X \in \mathcal{COP}^n$ is irreducible (w.r.t. $\mathcal{PSD}^n + \mathcal{N}^n$) if $\nexists A \in (\mathcal{PSD}^n + \mathcal{N}^n) \setminus \{O\}$ such that $X - A \in \mathcal{COP}^n$.

$X \in \mathrm{Ext}(\mathcal{COP}) \setminus (\mathcal{PSD}^n + \mathcal{N}^n) \;\Rightarrow\; X$ irreducible $\;\Rightarrow\; X \in \mathrm{bd}\,\mathcal{COP}$.

Contents

Main result

Theorem

For graph G have that Z_G is irreducible if and only if G is connected, α-critical and α-covered.

Proof uses results from [D. and Hildebrand, 2016].

Definition ([Small, 2015],[Erdös and Gallai, 1961],[Plummer, 1967],[Zykov, 1949])

G is α-critical if $\alpha(G - ij) > \alpha(G)$ for all $ij \in E(G)$.

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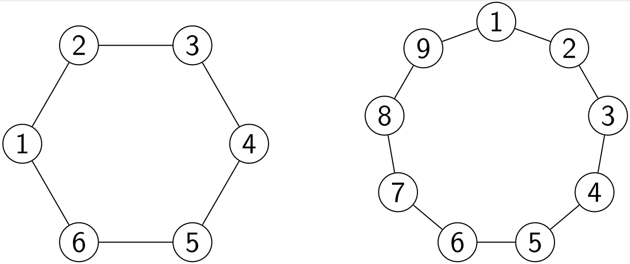

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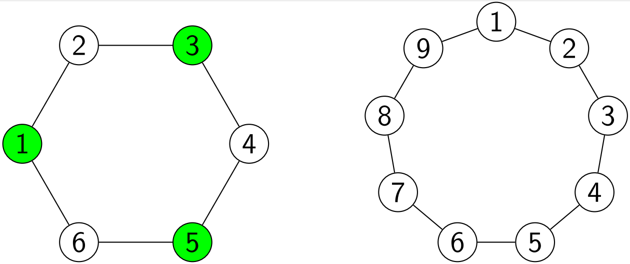

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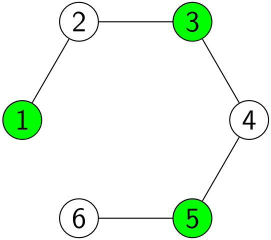

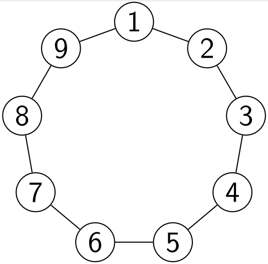

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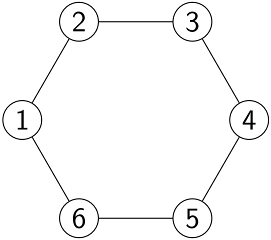

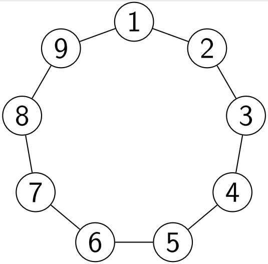

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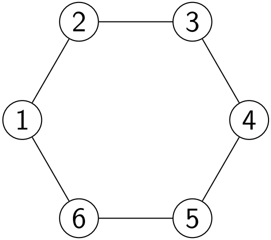

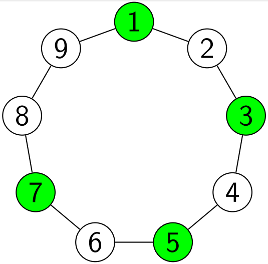

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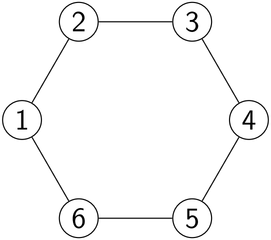

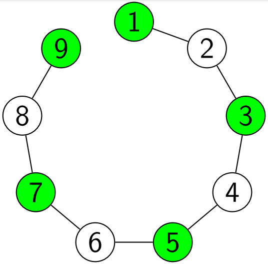

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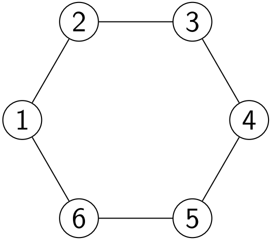

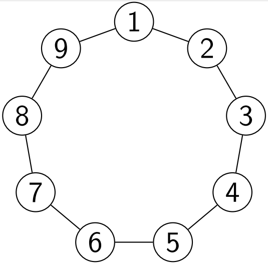

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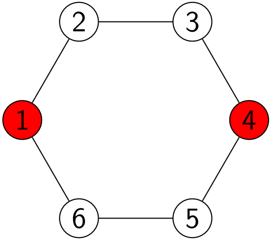

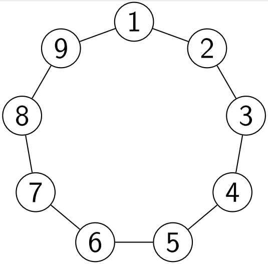

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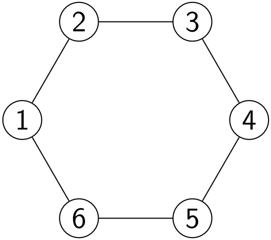

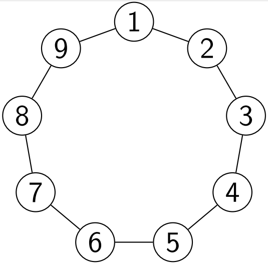

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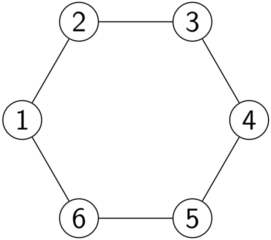

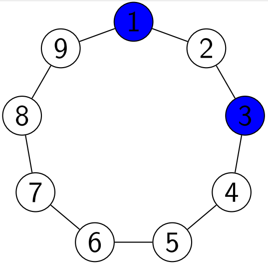

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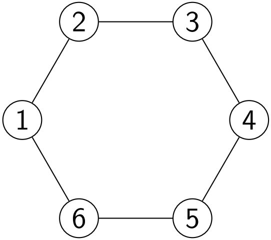

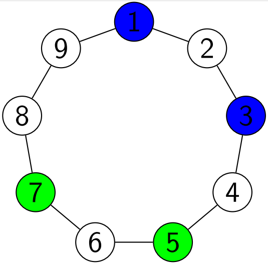

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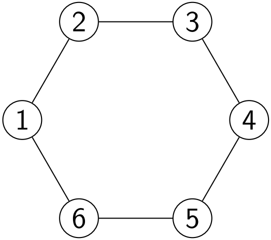

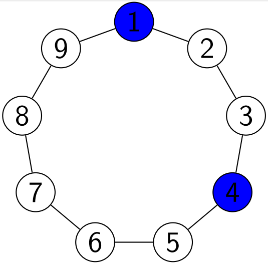

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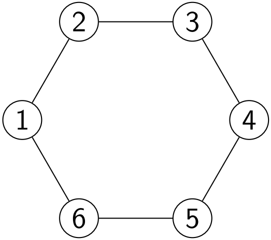

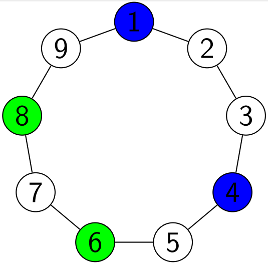

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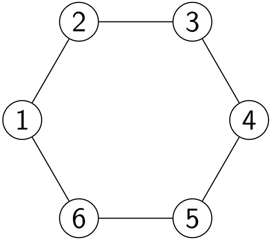

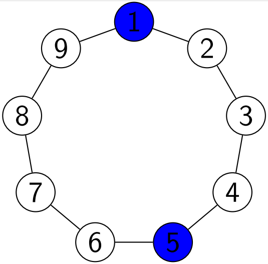

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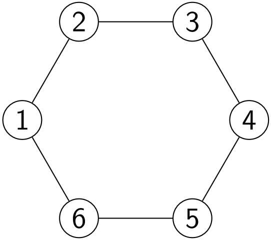

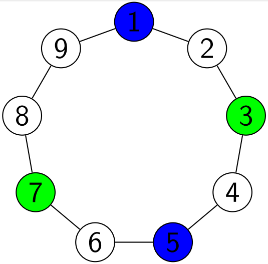

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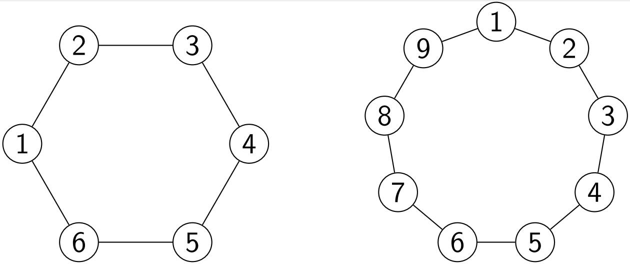
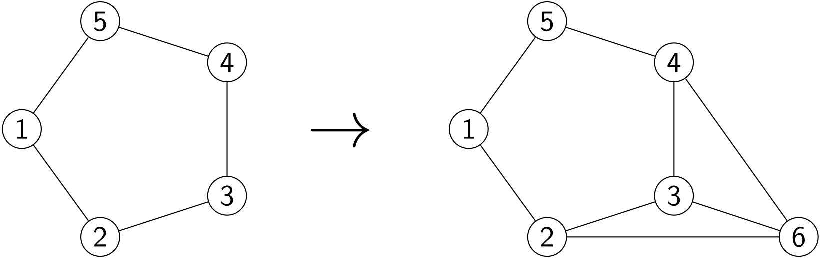


Contents

Vertex Duplication

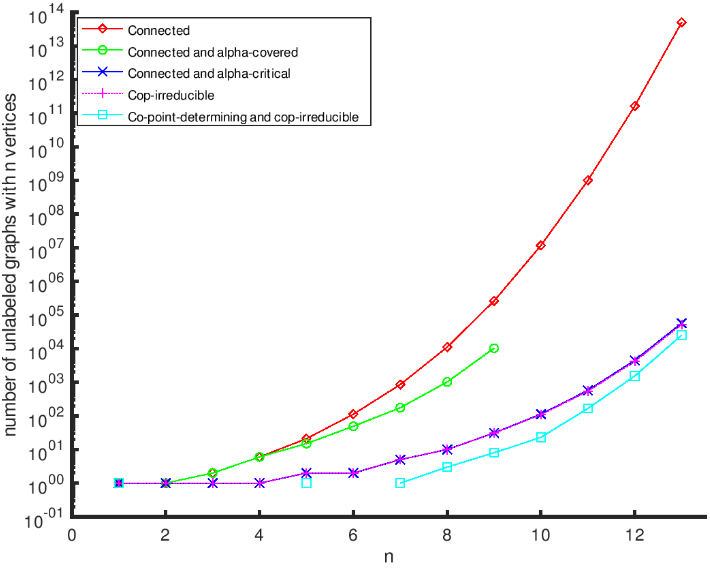


Lemma

Consider graphs G and H such that H is produced from G by duplicating one of its vertices.

$$\text{Then } G \text{ is } \left\{ \begin{array}{l} \text{connected} \\ \alpha\text{-critical} \\ \alpha\text{-covered} \\ \text{cop-irreducible} \end{array} \right\} \text{ if and only if } H \text{ is.}$$

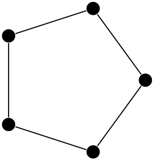
Counts



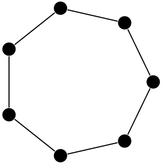
Connected
Connected and alpha-covered
Connected and alpha-critical
Cop-irreducible
Co-point-determining and cop-irreducible
number of unlabeled graphs with n vertices
n

Examples with $n \leq 9$

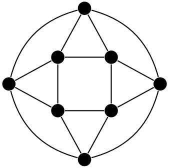
$n = 1,\ \alpha(G) = 1$



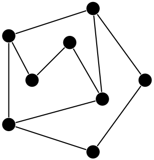
$n = 5,\ \alpha(G) = 2$



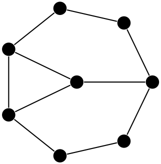
$n = 7,\ \alpha(G) = 3$



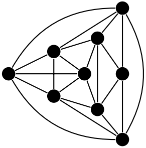
$n = 8,\ \alpha(G) = 2$



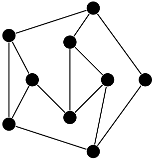
$n = 8,\ \alpha(G) = 3$



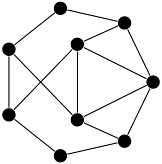
$n = 8,\ \alpha(G) = 3$



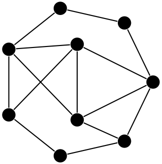
$n = 9,\ \alpha(G) = 2$



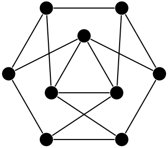
$n = 9,\ \alpha(G) = 3$



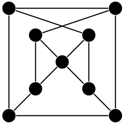
$n = 9,\ \alpha(G) = 3$



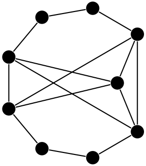
$n = 9,\ \alpha(G) = 3$



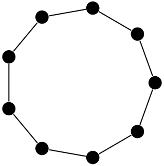
$n = 9,\ \alpha(G) = 3$



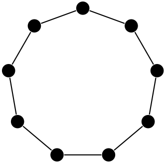
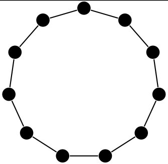
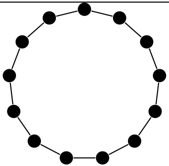
$n = 9,\ \alpha(G) = 3$

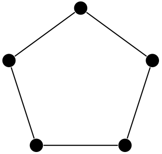
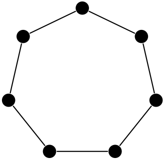
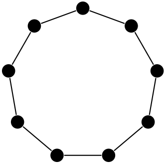
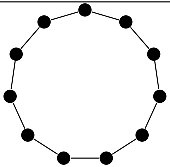
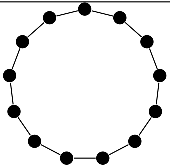


$n = 9,\ \alpha(G) = 3$

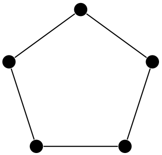
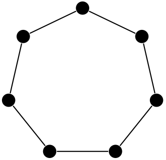
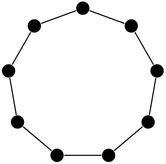
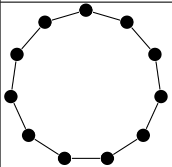
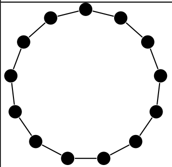
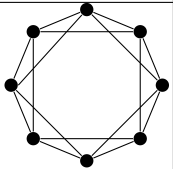
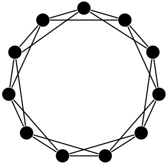
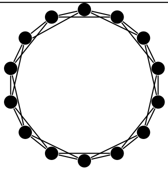
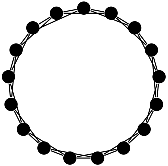
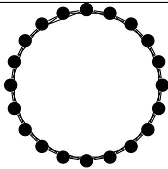
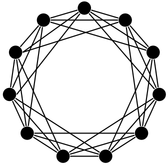
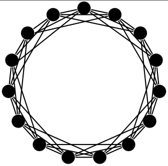
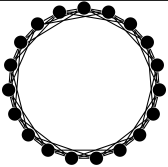
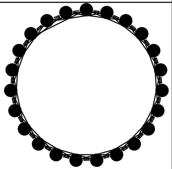
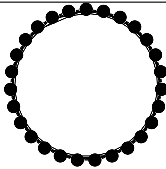


$n = 9,\ \alpha(G) = 4$

			Not Extreme		
			$\alpha = 4$	$\alpha = 5$	$\alpha = 6$
$n = 2\alpha + 1$					

	Extreme		Not Extreme		
	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$	$\alpha = 6$
$n = 2\alpha + 1$					

Extreme

	Extreme		Not Extreme		
	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$	$\alpha = 6$
$n = 2\alpha + 1$					
$n = 3\alpha + 2$					
$n = 4\alpha + 3$					

Thank You!

P.J.C. Dickinson and R. de Zeeuw, *Generating irreducible copositive matrices using the stable set problem*, Submitted

Slides of talk (+ bibliography) at `http://dickinson.website`.

Open problems at `http://open.dickinson.website`.

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